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A TAMED IMPLICIT-EXPLICIT SCHEME FOR STOCHASTIC CAHN-HILLIARD EQUATION DRIVEN BY MULTIPLICATIVE NOISE*

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Abstract. A fully discrete implicit-explicit scheme for stochastic Cahn-Hilliard equation driven by multiplicative noise in a two-dimensional setting is considered in this paper. The spatial discretization is a polynomial based spectral method and the temporal discretization is a tamed semi-implicit scheme which treats the nonlinear term explicitly. We show that the scheme is unconditionally stable under various norms, and establish optimal strong convergence rates. We also present numerical experiments to validate our theoretical results.

Keywords. Stochastic PDE; Cahn-Hilliard; full discretization; convergence rate.

AMS subject classifications. 65N35; 65E05; 65N12; 41A10; 41A25; 41A30; 41A58.

1. Introduction

We consider the following Cahn-Hilliard equation perturbed by multiplicative noise

$$\begin{cases} du = A\mu dt + G(u)dW^Q(t),\ x\in\mathcal{O}\subset\mathbb{R}^2,\\ \mu = -Au + F(u),\\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \mu}{\partial n} = 0, x\in\partial\mathcal{O},\\ u(0,\cdot) = u_0(x), \end{cases} \tag{1.1}$$

in a separable Hilbert space $H(\mathcal{O})$ equipped with norm $\|\cdot\|$, where A is the Neumann Laplacian operator, and F and G are Nemytskii operators defined by $F(u)(\xi) := f(u(\xi)) = u(\xi)^3 - u(\xi), G(u)(\xi) := g(u(\xi))$, $\xi\in\mathcal{O}$ respectively, $W^Q(t)$ is the Q-Wiener process on the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge 0},\mathbf{P})$ defined by (cf. [20])

$$W^Q(t) = \sum_{j=1}^{\infty}\sqrt{q_j}e_j\beta_j(t),$$

where $\beta_j(t)$ are independent standard Wiener processes, $\{e_j\}_{j=1}^{\infty}$ are orthonormal basis of $L^2(\mathcal{O})$, which are not necessarily eigenfunctions of A in $\mathcal{O}$. Note that if the noise is additive, i.e. $g(u) = I$, the Equation (1.1) reduces to the well-known Cahn-Hilliard-Cook (CHC) equation.

We recall that for $p\ge 1$ and some $\gamma\in(0,3/2)$ depending on smoothness of noise, (1.1) admits a unique mild solution (cf. [2])

$$u(t) = e^{-tA^2}u_0 - \int_0^t e^{-(t-s)A^2}Af(u(s))ds + \int_0^t e^{-(t-s)A^2}g(u)dW^Q(s),\ a.s. \tag{1.2}$$

provided that $u_0(x)\in H^\gamma(\mathcal{O})$. Furthermore, $\mathbf{E}\sup_{t\in[0,T]}\|u(t)\|_{H^\gamma}^p \le C$ (cf. [4,5]).

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The main goals of this work are (i) to study the spatial regularity of (1.2), (ii) to propose an unconditionally stable tamed scheme with a spectral approximation in space for (1.1); and (iii) to carry out a stability and error analysis of the full discretized scheme.

We now briefly describe our approach and main results. First, we note that optimal spatial regularity of stochastic CHC equation has been obtained in [25]. Following a similar approach, our first step is to show that the solution $u\in L^p(\Omega;C([0,T],H^\gamma))$ under Assumptions 2.1-2.2 below (see Theorem 2.1), where γ is the largest constant that makes these assumptions fulfilled. Applying $(-A)^{\frac{\gamma}{2}}$ on both sides of (1.2), one easily obtains

$$\begin{aligned}\mathbf{E}\|(-A)^{\frac{\gamma}{2}}u(t)\|^2 \le 3\left\|(-A)^{\frac{\gamma}{2}}e^{-tA^2}u_0\right\|^2+3\mathbf{E}\left\|(-A)^{\frac{\gamma}{2}}\int_0^t e^{-(t-\sigma)A^2}Af(u(\sigma))d\sigma\right\|^2\\ +3\mathbf{E}\left\|(-A)^{\frac{\gamma}{2}}\int_0^t e^{-(t-\sigma)A^2}g(u(\sigma))dW^Q(\sigma)\right\|^2.\end{aligned}$$

The bound of the first term is trivial if u_0 is sufficiently smooth and that of the second term can be derived by divide and conquer technique. The bound of the last one heavily relies on generalized Gronwall's inequality so as to ease a regularity restriction of g in Assumption 2.1, which is consistent with its additive counterpart ($\|(-\Delta)^{\frac{\gamma-2}{2}}Q^{\frac{1}{2}}\|_{HS}<\infty$).

In the aspect of temporal discretization, an essential difficulty in designing efficient algorithms for stochastic Cahn-Hilliard equation resides in its nonlinear term: an implicit treatment leads to a nonlinear system to solve at each time step, while an explicit treatment may lead to stability issues and also significantly increases the difficulties in analysis. A tamed method, which treats the nonlinear term explicitly, has been used for solving stochastic ODEs with locally Lipschitz conditions (cf. [9,26]), and more recently, for solving stochastic Allen-Cahn equations (cf. [8,24]). Following this manner, we propose the following tamed time discretization for (1.1):

$$\begin{cases}u^{n+1}-u^n=\tau\Delta\mu^{n+1}+g(u^n)\Delta W^n,\\ \mu^{n+1}=-\Delta u^{n+1}+\dfrac{1}{(1+\tau\|\Delta f(u^n)\|^2)}f(u^n),\end{cases}\tag{1.3}$$

where $\Delta W^n=W^Q(t_{n+1})-W^Q(t_n)\sim N(0,Q\tau)$. The above scheme only requires solving a linear equation with constant coefficients at each time step for each sample, on the other hand, fully implicit schemes, which require solving a nonlinear equation at each time step for each sample, are widely used in literature [5,15,17,25].

The spatial discretization for (1.3) is based on a Legendre-Galerkin method [21] with a set of specially constructed Fourier-like discrete eigenfunctions of A (cf. [21]), which are mutually orthogonal in both $L^2(\mathcal{O})$ and $H^1(\mathcal{O})$, as basis functions. Coupled with the tamed scheme (1.3), our full discretized scheme is fully decoupled as the widely used Fourier spectral method (cf. [3,5,11,12,24] etc), but has the following advantages:

- For problems with non-periodic boundary conditions, the Legendre-Galerkin method can achieve faster convergence rate than the Fourier spectral method for solutions with higher regularity, while Fourier spectral approximation may have certain advantages for SPDEs with lower spatial regularity (cf. [3,11,12,24]).
- Using the Legendre-Galerkin method can circumvent the commutativity restriction between the elliptic operator A and the covariance operator Q of $W^Q(t)$

in (1.1) pointed out by Jentzen, Kloeden and Winkel [11]: *The eigenfunctions of the dominating linear operator and of the covariance operator of the deriving additive noise process of the SPDE must coincide and must be known explicitly.*

Let $\{u_N^k\}$, where N is the highest degree of polynomials in the Legendre-Galerkin method, be the solution of our fully discretized scheme (see (3.2) below), we shall show that, under usual assumptions on g and u_0, (3.2) is unconditionally stable in the sense that

$$\mathbf{E}\left[\sup_{0\le k\le M-1}\|u_N^k\|_\infty^q\right]\le C(T,q,\|u_0\|), \qquad u_0\in H^\gamma(\mathcal{O}),\gamma\ge 2, \tag{1.4}$$

where C is a constant independent of M and N (see Theorem 3.3). This result controls the growth of nonlinear term and also provides an essential ingredient for the convergence analysis. With this stability result in hand, we shall prove the following strong convergence result (Theorem 4.3):

$$\mathbf{E}\|u(t_n)-u_N^n\|^2\le C(\tau+N^{-2\gamma}), \quad \gamma\ge 2. \tag{1.5}$$

We point out that this error analysis is valid for $d=2$ but requires a more restrictive stability result, i.e. Theorem 3.2-Theorem 3.4, compared with the tamed Legendre-Galerkin scheme for stochastic Allen-Cahn equations (cf. [8]).

In summary, the main contributions of this work are as follows:

- We propose an implicit-explicit unconditionally stable fully discretized scheme for (1.1). In each time step, we only need to solve an elliptic equation with constant coefficients. Furthermore, by using suitable basis functions, our fully discretized scheme can be fully decoupled;
- We establish optimal convergence rate in both space and time by exploring regularity of the exact solution $u(t)\in H^\gamma$ under usual assumptions;
- Our scheme avoids the commutativity restriction [11] on the elliptic operator A and covariance operator Q of $W^Q(t)$, which is assumed in most of existing works on spectral methods for stochastic Cahn-Hilliard or stochastic Allen-Cahn equation. Hence, its range of applications is significantly enlarged.

The outline of this paper is as follows. In Section 2, some preliminaries including our main assumptions and optimal spatial regularity for solutions of (1.1) are presented. In Section 3, we present our fully discretized scheme and show it to be unconditionally stable in various norms. In Section 4, we derive an optimal convergence result for our fully discrete scheme under suitable assumptions. We present in Section 5 an efficient implementation of our scheme followed by some numerical results to validate our scheme and theoretical results.

2. Preliminaries

In this section, we present some key ingredients for our algorithm development and convergence analysis.

2.1. Spaces and norms.

We begin with notations. Denote $H=L_2(\mathcal{O})$ with normal inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and

$$\dot{H}=\left\{v\in H:\int_{\mathcal{O}} v dx=0\right\}.$$

Let P be the orthogonal projector from H to $\dot{H}$. Then

$$(I-P)v=|\mathcal{O}|^{-1}\int_{\mathcal{O}}vdx$$

is the average of v.

Since A is a negative definite, self-adjoint, unbounded, linear operator on $\dot{H}$ with compact inverse, it generates an analytic semigroup $E(t)=e^{-tA^2},t\geq 0$ on $\dot{H}$. Moreover, $-A$ has an orthonormal basis with corresponding eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ satisfying

$$0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots\lambda_j\leq\cdots,\quad \lambda_j\to\infty.$$

In particular, the first eigenfunction is a constant, which is $|\mathcal{O}|^{-1/2}$ (cf. [15]). Consequently, the fractional powers of $(-A)$ and its domain $H^r:=\mathrm{dom}((-A)^{\frac{r}{2}})$ for all $r\in\mathbb{R}$ can be defined. In particular, $(-A)^r v=\sum_{j=1}^{\infty}\lambda_j^r(v,e_j)e_j, v\in\mathrm{dom}(-A)^r$. Thus, the associated semi-norm is given by

$$|v|_r=\|(-A)^{\frac{r}{2}}v\|=\left(\sum_{j=0}^{\infty}\lambda_j^r|(v,e_j)|^2\right)^{\frac{1}{2}}. \tag{2.1}$$

Then, the following inequalities hold (cf. [18, Theorem 6.13], [25]).

(i) For any $r\geq 0$, it holds that

$$(-A)^rE(t)x=E(t)(-A)^rx,\quad \text{for } x\in H^{2r},$$

and there exists a constant C such that

$$\|(-A)^rE(t)\|\leq Ct^{-\frac{r}{2}},\quad t>0; \tag{2.2}$$

(ii) For any $0\leq r\leq 2$, there exists a constant C such that

$$\|(-A)^{-r}(E(t)-I)\|\leq Ct^{\frac{r}{2}},\ t>0. \tag{2.3}$$

When the domain of A is extended to H, it is understood that for $v\in H$

$$Av=APv, \text{ and}$$
$$E(t)v=e^{-tA^2}v=\sum_{j=1}^{\infty}e^{-t\lambda_j^2}(v,e_j)e_j+(v,e_0)e_0=Pe^{-tA^2}v+(I-P)v. \tag{2.4}$$

Let V_N be the space of polynomials with degree $\leq N$. We define $\dot{V}_N=PV_N$, that is,

$$\dot{V}_N=\left\{v_N\in V_N:\int_{\mathcal{O}}v_Ndx=0\right\}$$

and we define the "discrete Laplacian" $A_N:V_N\to V_N$ by

$$-(A_Nv_N,w_N)=(\nabla v_N,\nabla w_N),\ v_N\in V_N,w_N\in V_N.$$

Clearly, A_N is negative definite, self-adjoint on $\dot{V}_N$ and its eigenvalues $\{-\lambda_{j,N}\}_{j=0}^{N}$ satisfy

$$0=\lambda_{0,N}<\lambda_{1,N}\leq\lambda_{2,N}\leq\cdots\leq\lambda_{N,N}.$$

Let $P_N: H \to V_N$ be the standard L^2 projection, so is $P_N: \dot{H} \to \dot{V}_N$.

Let U and V be two Hilbert spaces. We denote the norm in $L^p(\Omega,\mathcal{F},\mathbf{P};U)$ by $\|\cdot\|_{L^p(\Omega;U)}$, that is,

$$\|Y\|_{L^p(\Omega;U)} = \left(\mathbf{E}\left[\|Y\|_U^p\right]\right)^{\frac{1}{p}},\ Y \in L^p(\Omega,\mathcal{F},\mathbf{P};U).$$

Denote by $\mathcal{L}_1(U,V)$ the nuclear operator space from U to V and for $T \in \mathcal{L}_1(U,V)$, its norm is given by

$$\|T\|_{\mathcal{L}_1} = \sum_{i=1}^{\infty} |(Te_i,e_i)_U| \text{ and } Tr(T) = \sum_{i=1}^{\infty}(Te_i,e_i)_U$$

for any orthonormal basis $\{e_i\}$ of U. In this work, we assume that $W^Q(t)$ is of trace class, i.e. $Tr(Q) < \infty$. Let $\mathcal{L}_2(U,V)$ be the Hilbert-Schmidt space such that for any $T \in \mathcal{L}_2(U,V)$

$$\|T\|_{HS} = \|T\|_{\mathcal{L}_2(U,V)} = \left(\sum_{i=1}^{\infty} \|Te_i\|^2\right)^{1/2} < \infty.$$

Following [16], we introduce the space $U_0 := Q^{\frac{1}{2}}(U)$ together with its inner product

$$(u_0,v_0)_{U_0} = (Q^{-\frac{1}{2}}u_0, Q^{-\frac{1}{2}}v_0)_U, \quad \forall u_0,v_0 \in U_0.$$

Furthermore, we introduce the notations $\mathcal{L}_2^0 := \mathcal{L}_2(U_0,H)$ with its norm

$$\|T\|_{\mathcal{L}_2^0} = \|TQ^{\frac{1}{2}}\|_{\mathcal{L}_2(U,H)}, \qquad \forall T \in \mathcal{L}(U,H)$$

and $\mathcal{L}_{2,r}^0 = \mathcal{L}_2(U_0,H^r)$ with its norm

$$\|T\|_{\mathcal{L}_{2,r}^0} = \|(-A)^{\frac{r}{2}}T\|_{\mathcal{L}_2^0}.$$

REMARK 2.1. Since G is a Nemytskii operator induced by g and $W^Q(t)$ is of trace-class, one can have

$$\|G(u)\|^2_{\mathcal{L}_2^0} = \|g(u)Q^{\frac{1}{2}}\|^2_{\mathcal{L}^2} = \sum_{j=1}^{\infty}(g(u)Q^{\frac{1}{2}},e_j)^2 = \sum_{j=1}^{\infty} q_j(g(u),e_j)^2 \le Tr(Q)\|g(u)\|^2. \tag{2.5}$$

We shall start with the Burkholder-Davis-Gundy inequalilty for a sequence of H-valued discrete maringales (cf. [10,14]).

LEMMA 2.1. *Let $p \ge 2$ and $\{Z_m\}$ be a sequence of H-valued random variables with bounded p-moments such that $\mathbf{E}[Z_{m+1}|Z_0,\cdots,Z_m] = 0$ for all $1 \le m \le N-1$. Then there exists a constant $C = C(p)$ such that*

$$\left(\mathbf{E}\left\|\sum_{i=0}^{m} Z_i\right\|^p\right)^{\frac{1}{p}} \le C\left(\sum_{i=0}^{m}(\mathbf{E}\|Z_i\|^p)^{\frac{2}{p}}\right)^{\frac{1}{2}}. \tag{2.6}$$

2.2. Inequalities. The following lemmas are indispensable ingredients for our stability and convergence analysis.

LEMMA 2.2 (Generalized Gronwall inequality [7]). *Let $T>0$ and $C_1,C_2\geq 0$ and let $u(t)$ be a nonnegative and continuous function and $\beta>0$. If the following inequality holds*

$$u(t)\leq C_1+C_2\int_0^t (t-s)^{\beta-1}u(s)ds, \tag{2.7}$$

then there exists a constant $C=C(T,C_2,\beta)$ such that

$$u(t)\leq CC_1. \tag{2.8}$$

LEMMA 2.3 (Discrete generalized Gronwall inequality [7]). *For $\tau>0$, denote $t_j=j\tau, j=1,\cdots,M$ with $M\tau=T$. Let $C_1,C_2\geq 0$ and let $\{\phi_j\}_{j=1}^M$ be a sequence of nonnegative numbers. If for $\beta\in(0,1]$, we have*

$$\phi_j\leq C_1+C_2\tau\sum_{i=1}^{j-1}t_{j-i}^{\beta-1}\phi_i. \tag{2.9}$$

Then,

$$\phi_j\leq C(C_1,C_2,T,\beta).$$

LEMMA 2.4. *Let A_n,B_n,r_n be sequences of nonnegative numbers with $A_0=B_0=0$. For given $C_0,\tau>0$, if for each k,*

$$A_{k+1}^2-(1+C_0\tau)A_k^2+|A_{k+1}-A_k|^2+\frac{7\tau}{4}B_{k+1}^2\leq\tau r_k^2+\frac{\tau}{2}B_k^2, \tag{2.10}$$

then

$$A_{n+1}^2+\tau\sum_{k=1}^n B_k^2\leq\tau\sum_{k=0}^n(1+C_0\tau)^{n-k}r_k^2. \tag{2.11}$$

Proof. Lowering the index k in (2.10) by 1, we find

$$A_k^2-(1+C_0\tau)A_{k-1}^2+|A_k-A_{k-1}|^2+\frac{7\tau}{4}B_k^2\leq\tau r_{k-1}^2+\frac{\tau}{2}B_{k-1}^2. \tag{2.12}$$

Multiplying (2.12) by $(1+C_0\tau)$ and summing up with (2.10) gives

$$\begin{aligned}&A_{k+1}^2-(1+C_0\tau)^2A_{k-1}^2+\sum_{l=0}^1(1+C_0\tau)^l|A_{k-l+1}-A_{k-l}|^2+\frac{7\tau}{4}B_{k+1}^2\\&+\left[\frac{7\tau}{4}(1+C_0\tau)-\frac{\tau}{2}\right]B_k^2\leq\tau\sum_{l=0}^1(1+C_0\tau)^l r_{k-l}^2+\frac{\tau}{2}B_{k-1}^2.\end{aligned} \tag{2.13}$$

Repeating this process for $k=1$ to n, we can arrive at

$$\begin{aligned}&A_{n+1}^2+\sum_{k=0}^n(1+C_0\tau)^k|A_{n-k+1}-A_{n-k}|^2+\frac{7\tau}{4}B_{n+1}^2\\&+\sum_{k=1}^n\left[\frac{7\tau}{4}(1+C_0\tau)-\frac{\tau}{2}\right](1+C_0\tau)^{n-k}B_k^2\leq\tau\sum_{k=0}^n(1+C_0\tau)^k r_{n-k}^2,\end{aligned} \tag{2.14}$$

which implies the desired result. ∎

LEMMA 2.5 ([8]). *There exist positive constants c_1 and c_2 independent of N and τ such that*

$$c_1\|(-\Delta_N)^\gamma v_N\| \le \|(-\Delta)^\gamma v_N\| \le c_2\|(-\Delta_N)^\gamma v_N\|, \quad -\frac{1}{2}\le\gamma\le\frac{1}{2}. \tag{2.15}$$

2.3. Spatial regularity of $u(t)$. We first describe some essential assumptions.

ASSUMPTION 2.1 (Linear growth and Lipschitz condition). *The mapping $g(v)$ satisfies*

$$\|g(u)\|_{\mathcal{L}^0_{2,\nu}} \le c(1+\|u\|_\nu), \quad u\in H^\nu(\mathcal{O}), \ (\nu=0,1,\gamma-2+\epsilon,\gamma)$$

and

$$\|g(u)-g(v)\|_{\mathcal{L}^0_{2,\nu}} \le c\|u-v\|_\nu, \ u,v\in H^\nu(\mathcal{O}), \ (\nu=-1,0).$$

ASSUMPTION 2.2 (Initial condition). *The initial condition u_0 is $\mathscr{F}_0/\mathscr{B}(H^\gamma)$-measurable and*

$$\mathbf{E}\|u_0\|_\gamma^p \le C, \quad p\ge 2.$$

We now proceed to exploit the regularity of (1.2) under these assumptions. Note that various spatial regularities have been established under the conditions $\|(-A)^{\frac{\gamma-2}{2}}Q^{\frac{1}{2}}\|_{HS}<\infty$ (cf. [15, 25]) for additive noise case.

THEOREM 2.1. *Let Assumptions 2.1-2.2 be fulfilled and let $\mathcal{O}$ be a bounded convex spatial domain with smooth boundary. Then, for any $p\ge 2$, the unique mild solution $u(t)$ of (1.1) satisfies*

$$\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|_\gamma^p < C, \quad \gamma>0,$$

where γ is the largest constant allowed by Assumptions 2.1 and 2.2.

Furthermore, $\sup_{t\in[0,T]} \mathbf{E}\|f(u(t))\|_{L^p(\Omega;H^\gamma)} < C$ provided $\gamma d>1$.

Proof. We start with (1.2). For any $t>0$,

$$\begin{aligned}\|u(t)\|_{L^p(\Omega;H^\gamma)} \le& \left\|(-A)^{\frac{\gamma}{2}}E(t)u_0\right\|_{L^p(\Omega;H)} + \left\|(-A)^{\frac{\gamma}{2}}\int_0^t E(t-\sigma)Af(u(\sigma))d\sigma\right\|_{L^p(\Omega;H)}\\ &+\left\|(-A)^{\frac{\gamma}{2}}\int_0^t E(t-\sigma)g(u(\sigma))dW^Q(\sigma)\right\|_{L^p(\Omega;H)}.\end{aligned} \tag{2.16}$$

The assumption on $u_0:\Omega\to H^\gamma$ implies the bound for the first term

$$\left\|E(t)(-A)^{\frac{\gamma}{2}}u_0\right\|_{L^p(\Omega;H)} \le \|u_0\|_{L^p(\Omega;H^\gamma)} < C. \tag{2.17}$$

For the last term in (2.16), we have by Assumption 2.1

$$\left\|\int_0^t (-A)^{\frac{\gamma}{2}}E(t-\sigma)g(u(\sigma))dW^Q(\sigma)\right\|_{L^p(\Omega;H)}$$

$$
\begin{aligned}
&\le C(p)\Big(\int_0^t \big(\mathbf{E}\big\|(-A)^{\frac{\gamma}{2}}E(t-\sigma)g(u(\sigma))\big\|^p_{\mathcal{L}_2^0}\big)^{\frac{2}{p}}d\sigma\Big)^{\frac12}\\
&\le C(p)\Big(\int_0^t \big(\mathbf{E}\big\|(-A)^{\frac{2-\epsilon}{2}}E(t-\sigma)(-A)^{\frac{\gamma-2+\epsilon}{2}}g(u(\sigma))\big\|^p_{\mathcal{L}_2^0}\big)^{\frac{2}{p}}d\sigma\Big)^{\frac12}\\
&\le C\Big(\int_0^t (t-\sigma)^{\frac{\epsilon-2}{2}}\big(\mathbf{E}\|g(u(\sigma))\|^p_{\mathcal{L}^0_{2,\gamma-2+\epsilon}}\big)^{\frac{2}{p}}d\sigma\Big)^{\frac12}\\
&\le C\Big(\int_0^t (t-\sigma)^{\frac{\epsilon-2}{2}}\big(C+\mathbf{E}\|(u(\sigma))\|^p_{\mathcal{L}^0_{2,\gamma-2+\epsilon}}\big)^{\frac{2}{p}}d\sigma\Big)^{\frac12}\\
&\le C\Big(\int_0^t (t-\sigma)^{\frac{\epsilon-2}{2}}\|u(\sigma)\|^2_{L^p(\Omega;H^\gamma)}d\sigma\Big)^{\frac12}+C,
\end{aligned}\tag{2.18}
$$

where $\epsilon>0$ is arbitrarily small.

Therefore, we only need to bound the second term in (2.16). To this end, we consider γ in different intervals and conquer each in turn. The case $\gamma\in(0,1]$ has been covered in [4,5].

(i) Case $\gamma\in(1,2)$,

$$
\Big\|(-A)^{\frac{\gamma}{2}}\int_0^t E(t-\sigma)Af(u(\sigma))d\sigma\Big\|_{L^p(\Omega;H)}\le\int_0^t(t-\sigma)^{-\frac{\gamma+2}{4}}\|f(u(\sigma))\|_{L^p(\Omega;H)}d\sigma<C.\tag{2.19}
$$

Therefore, $u(t)\in L^p(\Omega;H^\gamma)$ by the generalized Gronwall's inequality (Lemma 2.2) using (2.17),(2.19) and (2.18). Since $\gamma>1$, we have $H^\gamma(\mathcal{O})$ is a Banach algebra for $d=1$ or 2 (cf. [1, Page 106]). Hence, $\sup_t \mathbf{E}\|f(u(t))\|^p_\gamma<\infty$.

(ii) Case $\gamma=2$.

From the previous case, one has $u(t)\in L^p(\Omega;H^{\frac74})$ and $\sup_t\mathbf{E}\|f(u(t))\|^p_{\frac74}<\infty$. Therefore,

$$
\begin{aligned}
\Big\|\int_0^t A^2E(t-s)f(u(s))ds\Big\|&=\Big\|\int_0^t(-A)^{\frac98}E(t-s)(-A)^{\frac78}f(u(s))ds\Big\|\\
&\le C\int_0^t(t-s)^{-\frac{9}{16}}\|f(u(s))\|_{L^p(\Omega;H^{\frac74})}ds<C.
\end{aligned}\tag{2.20}
$$

Therefore, $u(t)\in L^p(\Omega;H^2)$ and $\sup_t\mathbf{E}\|f(u(t))\|^p_2<\infty$ by the similar reason as in the previous case .

(iii) Case $\gamma\in(2,4)$.

By virtue of the results of case (ii),

$$
\Big\|\int_0^t(-A)^{\frac{\gamma}{2}}E(t-\sigma)Af(u(\sigma))d\sigma\Big\|_{L^p(\Omega;H)}\le\int_0^t(t-\sigma)^{-\frac{\gamma}{4}}\|f(u(\sigma)\|_{L^p(\Omega;H^2)}d\sigma<C.\tag{2.21}
$$

We repeat the above process for arbitrarily large γ as long as both (2.17) and (2.18) hold or Assumptions 2.1 and 2.2 hold.

The proof is completed. ◻

For later use, we present a local Lipschitz continuity for the nonlinear f.

LEMMA 2.6. *For $d=1,2$ and $\gamma>1$,*

$$\|(-A_N)^{-\frac{1}{2}}P_N(f(u)-f(v))\|\le C(1+\|u\|_1^2+\|v\|_1^2)\|u-v\|,\quad u,v\in H^1(\mathcal{O}); \tag{2.22}$$

$$\|(-A_N)^{\frac{1}{2}}P_N(f(u)-f(v))\|\le C(1+|u|_1^2+|v|_1^2+\|u\|_\infty^2+\|v\|_\infty^2)|u-v|_1,\ u,v\in H^1(\mathcal{O}). \tag{2.23}$$

Proof. The first inequality can be found in [15] and we only prove the second one. By using (2.15) and the embedding $H^1\hookrightarrow L^q$, where q is large

$$\begin{aligned}
&\|(-A_N)^{\frac{1}{2}}P_N(f(u)-f(v))\|\le C\|\nabla(f(u)-f(v))\|\\
&\le C\|\nabla(u-v)(u^2+uv+v^2)\|+C\|(u-v)\nabla(u^2+uv+v^2)\|\\
&\le C|u-v|_1(\|u\|_\infty^2+\|v\|_\infty^2)+C\|u-v\|_{L^q}\|u\nabla u+u\nabla v+v\nabla u+v\nabla v\|_{L^{\frac{q}{q-1}}}\\
&\le C|u-v|_1(\|u\|_\infty^2+\|v\|_\infty^2)+C|u-v|_1(|u|_1^2+|v|_1^2),
\end{aligned} \tag{2.24}$$

where we have computed

$$\|u\nabla v\|_{L^{\frac{q}{q-1}}}:=\|u\nabla v\|_{L^{1+\epsilon}}\le|v|_1\|u\|_\infty\le\frac{|v|_1^2}{2}+\frac{\|u\|_\infty^2}{2}. \tag{2.25}$$

∎

3. Stability analysis

We first discretize the scheme (1.3) in space using a mixed weak formulation as follows:

$$(u_N^{n+1}-u_N^n,\phi)=-\tau(\nabla\mu_N^{n+1},\nabla\phi)+(g(u_N^n)\Delta W^n,\phi),\quad\forall\phi\in\dot{V}_N \tag{3.1a}$$

$$(\mu_N^{n+1},\psi)=-(\Delta u_N^{n+1},\psi)+\frac{1}{(1+\tau\|\Delta f(u_N^n)\|^2)}(Pf(u_N^n),\psi),\ \forall\psi\in\dot{V}_N, \tag{3.1b}$$

where the discrete spaces are described in the last section. We shall establish an unconditional stability result for (3.1a)-(3.1b), which turns out to be more restrictive and more involved compared with its fully-implicit counterpart (cf. [5,15,17,25]).

Similar as [15,25], with A_N and P_N in our hands, we can rewrite the above equation into abstract form

$$u_N^{k+1}-u_N^k=-\tau A_N^2u_N^{k+1}-\frac{\tau A_NP_NPf(u_N^k)}{1+\tau\|\Delta f(u_N^n)\|^2}+P_N[g(u_N^k)\Delta W^k] \tag{3.2}$$

or

$$\begin{aligned}
u_N^n=E_N^nP_Nu_0-&\sum_{k=1}^n\int_{t_{k-1}}^{t_k}\frac{E_N^{n-k+1}A_NP_NPf(u_N^{k-1})}{1+\tau\|\Delta f(u_N^{k-1})\|^2}ds\\
&+\sum_{k=1}^n\int_{t_{k-1}}^{t_k}E_N^{n-k+1}P_N[g(u_N^{k-1})dW^Q(s)],
\end{aligned} \tag{3.3}$$

where $E_N^n=(I+\tau A_N^2)^{-n},1\le n\le M$, which is an approximation of $e^{-t_nA_N^2}$.

It is obvious that

$$\begin{aligned}
E_N^nP_Nv&=\sum_{j=0}^N(1+\tau\lambda_{j,N}^2)^{-n}(P_Nv,\phi_{j,N})\phi_{j,N}\\
&=\sum_{j=1}^N(1+\tau\lambda_{j,N}^2)^{-n}(P_Nv,\phi_{j,N})\phi_{j,N}+\frac{1}{|\Omega|}\int_\Omega vdx=PE_N^nP_Nv+(I-P)v.
\end{aligned} \tag{3.4}$$

Here, $\phi_{j,N}$ are eigenfunctions of $-A_N$.

Denote by $r(z):=(1+z)^{-1}$, and we can have $E^n=r(\tau A_N^2)^n$. Furthermore, from (cf. [22, Theorem 7.1]), there exist constants C and c such that

$$|r(z)-e^{-z}|\le Cz^2,\ |r(z)|\le e^{-cz},\ \ \forall z\in[0,1]. \tag{3.5}$$

Similar to [22, Lemma 3.9], one also has the following smoothing property

$$\|(-A_N)^{\mu}E_N^n y_N\|\le Ct_n^{-\frac{\mu}{2}}\|y_N\|,\quad y_N\in V_N. \tag{3.6}$$

We next prove our first unconditional stability property of the scheme.

THEOREM 3.1. *The scheme (3.1a)-(3.1b) is unconditionally stable in the sense that*

$$\mathbf{E}\sup_{0\le n\le M}\|u_N^n\|^{2m}\le C(\|u_0\|,T,K,m),\ \forall\ m\ge 1.$$

Proof. The proof is divided into two steps.

Step 1: Choosing $\phi=u_N^{n+1}-u_N^n$ in (3.1a) and $\psi=\tau\Delta(u_N^{n+1}-u_N^n)$ in (3.1b), and carrying out integration by parts lead to

$$\begin{aligned}
&\|u_N^{n+1}-u_N^n\|^2=-\tau(\nabla\mu_N^{n+1},\nabla(u_N^{n+1}-u_N^n))+(g(u_N^n)\Delta W^n,u_N^{n+1}-u_N^n),\\
&-\tau(\nabla\mu_N^{n+1},\nabla(u_N^{n+1}-u_N^n))=-\tau(\Delta u_N^{n+1},\Delta u_N^{n+1}-\Delta u_N^n)\\
&\qquad\qquad+\frac{\tau}{1+\tau\|\Delta f(u_N^n)\|^2}(\Delta f(u_N^n),u_N^{n+1}-u_N^n).
\end{aligned} \tag{3.7}$$

Adding up these two identities gives

$$\begin{aligned}
&\|u_N^{n+1}-u_N^n\|^2+\frac{\tau}{2}\|\Delta u_N^{n+1}\|^2+\frac{\tau}{2}\|\Delta u_N^{n+1}-\Delta u_N^n\|^2\\
=&\frac{\tau}{2}\|\Delta u_N^n\|^2+(g(u_N^n)\Delta W^n,u_N^{n+1}-u_N^n)+\frac{\tau}{1+\tau\|\Delta f(u_N^n)\|^2}(\Delta f(u_N^n),u_N^{n+1}-u_N^n)\\
\le&\frac{\tau}{2}\|\Delta u_N^n\|^2+\|g(u_N^n)\Delta W^n\|^2+\frac{1}{2}\|u_N^{n+1}-u_N^n\|^2+\frac{\tau^2\|\Delta f(u_N^n)\|^2}{1+\tau\|\Delta f(u_N^n)\|^2}.
\end{aligned} \tag{3.8}$$

Therefore,

$$\begin{aligned}
&\|u_N^{n+1}-u_N^n\|^2+\tau\|\Delta u_N^{n+1}\|^2+\tau\|\Delta u_N^{n+1}-\Delta u_N^n\|^2\\
\le&\tau\|\Delta u_N^n\|^2+2\|g(u_N^n)\Delta W^n\|^2+2\tau.
\end{aligned} \tag{3.9}$$

Summing over n implies

$$\tau\|\Delta u_N^{n+1}\|^2+\tau\sum_{j=0}^n\|\Delta u_N^{j+1}-\Delta u_N^j\|^2\le 2\sum_{j=0}^n\|g(u_N^j)\Delta W^j\|^2+2T+\tau\|\Delta u_N^0\|^2. \tag{3.10}$$

Taking m-th power and then expectation on both sides, we have

$$\begin{aligned}
&\mathbf{E}(\tau\|\Delta u_N^{n+1}\|^2)^m+\mathbf{E}\left(\tau\sum_{j=0}^n\|\Delta u_N^{j+1}-\Delta u_N^j\|^2\right)^m\\
\le& C\mathbf{E}\left(\sum_{j=0}^n\|g(u_N^j)\Delta W^j\|^2\right)^m+CT^m+C\mathbf{E}(\tau\|\Delta u_N^0\|^2)^m
\end{aligned}$$

$$
\begin{aligned}
&\le Cn^{m-1}\mathbf{E}\sum_{j=0}^{n}\|g(u_N^j)\Delta W^j\|^{2m}+C\\
&\le C\tau\mathbf{E}\sum_{j=0}^{n}\|g(u_N^j)\|^{2m}+C\le C\tau\sum_{j=0}^{M-1}\mathbf{E}\sup_{0\le j\le M-1}\|u_N^j\|^{2m}+C,
\end{aligned}\tag{3.11}
$$

where we have used Assumption 2.1 (with $\nu=0$) in the last step.

Step 2: Similarly, choosing $\phi=u_N^{n+1}$ in (3.1a) and $\psi=\tau\Delta u_N^{n+1}$ in (3.1b) implies

$$
\begin{aligned}
&\frac{1}{2}(\|u_N^{n+1}\|^2-\|u_N^n\|^2+\|u_N^{n+1}-u_N^n\|^2)=-\tau(\nabla\mu_N^{n+1},\nabla u_N^{n+1})+(g(u_N^n)\Delta W^n,u_N^{n+1}),\\
&-\tau(\nabla\mu_N^{n+1},\nabla u_N^{n+1})=-\tau\|\Delta u_N^{n+1}\|^2+\frac{\tau}{1+\tau\|\Delta f(u_N^n)\|^2}(Pf(u_N^n),\Delta u_N^{n+1}).
\end{aligned}\tag{3.12}
$$

Let us handle the drift term next. By the Gargliardo-Nirenberg inequality,

$$
\begin{aligned}
-\tau(\nabla f(u_N^n),\nabla u_N^n)&=\tau\|\nabla u_N^n\|^2-3\tau(u_N^n\nabla u_N^n,u_N^n\nabla u_N^n)\le\tau\|\nabla u_N^n\|^2\\
&\le C\tau\|u_N^n\|\|\Delta u_N^n\|\le C\tau\|u_N^n\|^2+\frac{\tau}{4}\|\Delta u_N^n\|^2\\
&\le C\tau\|u_N^n\|^2+\frac{\tau}{2}\|\Delta u_N^{n+1}\|^2+\frac{\tau}{2}\|\Delta u_N^{n+1}-\Delta u_N^n\|^2.
\end{aligned}\tag{3.13}
$$

Adding up the equations in (3.12), we have by Assumptions 2.1-2.2

$$
\begin{aligned}
&\frac{1}{2}(\|u_N^{n+1}\|^2-\|u_N^n\|^2+\|u_N^{n+1}-u_N^n\|^2)+\tau\|\Delta u_N^{n+1}\|^2\\
=&\frac{-\tau}{1+\tau\|\Delta f(u_N^n)\|^2}(\nabla f(u_N^n),\nabla u_N^{n+1})+(g(u_N^n)\Delta W^n,u_N^{n+1}),\\
=&\frac{-\tau}{1+\tau\|\Delta f(u_N^n)\|^2}(\nabla f(u_N^n),\nabla(u_N^{n+1}-u_N^n))-\frac{\tau}{1+\tau\|\Delta f(u_N^n)\|^2}(\nabla f(u_N^n),\nabla u_N^n)\\
&\quad+(g(u_N^n)\Delta W^n,u_N^{n+1}-u_N^n)+(g(u_N^n)\Delta W^n,u_N^n),\\
\le&\frac{\tau}{1+\tau\|\Delta f(u_N^n)\|^2}\left(\tau\|\Delta f(u_N^n)\|^2+\frac{1}{4\tau}\|(u_N^{n+1}-u_N^n)\|^2\right)+C\tau\|u_N^n\|^2+\frac{\tau}{2}\|\Delta u_N^{n+1}\|^2\\
&\quad+\frac{\tau}{2}\|\Delta u_N^{n+1}-\Delta u_N^n\|^2+\|g(u_N^n)\Delta W^n\|^2+\frac{1}{4}\|u_N^{n+1}-u_N^n\|^2+(g(u_N^n)\Delta W^n,u_N^n)\\
\le&\tau+\frac{1}{2}\|u_N^{n+1}-u_N^n\|^2+C\tau\|u_N^n\|^2+\frac{\tau}{2}\|\Delta u_N^{n+1}\|^2+\frac{\tau}{2}\|\Delta u_N^{n+1}-\Delta u_N^n\|^2\\
&\quad+\|g(u_N^n)\Delta W^n\|^2+(g(u_N^n)\Delta W^n,u_N^n).
\end{aligned}\tag{3.14}
$$

Denote by $B=1+C\tau$. Using the elementary inequality $1+x\le e^x,\ x\ge 0$ gives

$$
B^n=\left(1+\frac{CT}{M}\right)^n\le e^{CT},\ n\le M.\tag{3.15}
$$

Hence, summing over n in (3.14) leads to

$$
\begin{aligned}
\|u_N^{n+1}\|^2+\tau\sum_{j=0}^{n}B^{n-j}\|\Delta u_N^{j+1}\|^2\le& B^{n+1}\|u_N^0\|^2+2T\sum_{j=0}^{n}B^{n-j}+\tau\sum_{j=0}^{n}B^{n-j}\|\Delta u_N^{j+1}-\Delta u_N^j\|^2\\
&+\sum_{j=0}^{n}B^{n-j}\|g(u_N^j)\Delta W^j\|^2+\sum_{j=0}^{n}B^{n-j}(g(u_N^j)\Delta W^j,u_N^j).
\end{aligned}\tag{3.16}
$$

Next, we take m-th moment on both sides of (3.16) and obtain

$$\begin{aligned}
&\sup_{0\le n\le M-1}\|u_N^{n+1}\|^{2m}\\
\le& C\|u_N^0\|^{2m}+CT^m+C\left[\sum_{j=0}^{M-1}\|g(u_N^j)\Delta W^j\|^2\right]^m+C\left(\tau\sum_{j=0}^{M-1}\|\Delta u_N^{j+1}-\Delta u_N^j\|^2\right)^m\\
&+C\sup_{0\le n\le M-1}\left|\sum_{j=0}^{n}(g(u_N^j)\Delta W^j,u_N^j)\right|^m. \qquad (3.17)
\end{aligned}$$

Since $(g(u_N^j)\Delta W^j,u_N^j)$ are martingales independent from each other, we have by the discrete BDG inequality (Lemma 2.1),

$$\begin{aligned}
&\mathbf{E}\sup_{0\le n\le M-1}\left|\sum_{j=0}^{n}(g(u_N^j)\Delta W^j,u_N^j)\right|^m\\
\le& C\left(\sum_{j=0}^{M-1}[\mathbf{E}(|(g(u_N^j)\Delta W^j,u_N^j)|^m]^{\frac{2}{m}}\right)^{\frac{m}{2}}\\
\le& C\left(\sum_{j=0}^{M-1}\{\mathbf{E}[(\|g(u_N^j)\|^4+\|u_N^j\|^4)\tau]^{\frac{m}{2}}\}^{\frac{2}{m}}\right)^{\frac{m}{2}}\\
\le& C\tau^{\frac{m}{2}}\left(\sum_{j=0}^{M-1}(\mathbf{E}\sup_{0\le j\le M-1}\|u_N^j\|^{2m})^{\frac{2}{m}}\right)^{\frac{m}{2}}\\
\le& C\tau\sum_{j=0}^{M-1}\mathbf{E}\sup_{0\le j\le M-1}\|u_N^j\|^{2m}. \qquad (3.18)
\end{aligned}$$

Hence, (3.11) and (3.18) imply

$$\begin{aligned}
&\mathbf{E}\sup_{0\le n\le M-1}\|u_N^{n+1}\|^{2m}\le C\mathbf{E}\|u_N^0\|^{2m}+CT^m+C\mathbf{E}\left(\tau\sum_{j=0}^{M-1}\|\Delta u_N^{j+1}-\Delta u_N^j\|^2\right)^m\\
&\quad+C\mathbf{E}\left[\sum_{j=0}^{M-1}\|g(u_N^j)\Delta W^j\|^2\right]^m+C\mathbf{E}\sup_{0\le n\le M-1}\left|\sum_{j=0}^{n}(g(u_N^j)\Delta W^j,u_N^j)\right|^m\\
\le& C\mathbf{E}\|u_N^0\|^{2m}+CT^m+C\tau\sum_{j=0}^{M-1}\mathbf{E}\sup_{0\le j\le M-1}\|u_N^j\|^{2m}. \qquad (3.19)
\end{aligned}$$

Hence, the discrete Gronwall inequality implies

$$\mathbf{E}\sup_{0\le n\le M-1}\|u_N^{n+1}\|^{2m}\le C(\|u^0\|,T,K,m). \qquad (3.20)$$

Furthermore, we also have

$$\mathbf{E}\sup_{0\le n\le M-1}\|u_N^{n+1}\|^{2m}+\mathbf{E}\left(\sum_{j=0}^{M-1}\|\Delta u_N^{j+1}\|^2\tau\right)^m\le C(\|u_0\|,T,K,m). \qquad (3.21)$$

∎

We need the following lemma to prove a stronger stability result for the sake of convergence analysis.

LEMMA 3.1. *Let $v_j \in dom((-A_N)^{\frac{1}{2}})$. Then,*

$$\left\|\tau\sum_{j=1}^{n}(-A_N)^{\frac{3}{2}}E_N^{n-j+1}v_j\right\| \le C\left\|\sum_{j=1}^{n}(-A_N)^{\frac{1}{2}}v_j\tau\right\|; \tag{3.22}$$

$$\left\|\tau\sum_{j=1}^{n}(-A_N)^{2}E_N^{n-j+1}v_j\right\| \le C\sup_{1\le j\le n}\|v_j\|. \tag{3.23}$$

Proof. Let $\phi_{i,N}$ is a set of eigenfunctions of $-A_N$., which are orthonormal in L^2, and let $\lambda_{i,N}$ be its associated eigenvalue. Then, we first prove that

$$\tau\sum_{j=1}^{n}\lambda_{i,N}^2 r(\tau\lambda_{i,N}^2)^{2n+2-2j} \le C \tag{3.24}$$

by modifying some arguments from [25, Lemma 5.2]. For the case $\tau\lambda_{i,N}^2 \le 1$, an application of (3.5) gives

$$\begin{aligned}&\tau\sum_{j=0}^{k-1}\lambda_{i,N}^2 r(\tau\lambda_{i,N}^2)^{2k+2-2j} \le \tau\sum_{j=0}^{k-1}\lambda_{i,N}^2 e^{-2c(t_{k+1}-s)\lambda_{i,N}^2}\\ \le&\int_0^{t_{k-1}}\lambda_{i,N}^2 e^{-2c(t_{k-1}-s)\lambda_{i,N}^2}\,ds \le \frac{1}{2c};\end{aligned} \tag{3.25}$$

for the other case $\tau\lambda_{i,N}^2 > 1$, we easily have $r(\tau\lambda_{i,N}^2) < \frac{1}{2}$ and

$$\tau\sum_{j=0}^{k-1}\lambda_{i,N}^2 r(\tau\lambda_{i,N}^2)^{2k+2-2j} \le \sum_{j=0}^{k-1} r(\tau\lambda_{i,N}^2)^{2k+1-2j} < \sum_{j=0}^{k-1} 2^{-(2k+1-2j)} \le \frac{2}{3}. \tag{3.26}$$

Thus, (3.24) is validated. For the first inequality, we have

$$\begin{aligned}&\left\|\tau\sum_{j=1}^{n}(-A_N)^{\frac{3}{2}}E_N^{n-j+1}v_j\right\|^2 \le \sum_{i=1}^{N}\left|\tau\sum_{j=1}^{n}\lambda_{i,N} r(\tau\lambda_{i,N}^2)^{n+1-j}((-A_N)^{\frac{1}{2}}v_j,\phi_{i,N})\right|^2\\ \le&\sum_{i=1}^{N}\left(\tau\sum_{j=1}^{n}\lambda_{i,N}^2 r(\tau\lambda_{i,N}^2)^{2n+2-2j}\right)\left(\tau\sum_{j=1}^{n}((-A_N)^{\frac{1}{2}}v_j,\phi_{i,N})^2\right)\\ \le&C\left\|\sum_{j=1}^{n}(-A_N)^{\frac{1}{2}}v_j\tau\right\|^2.\end{aligned} \tag{3.27}$$

Similarly, from the proof of [25, Lemma 5.2],

$$\begin{aligned}&\left\|\tau\sum_{j=1}^{n}(-A_N)^2E_N^{n-j+1}v_j\right\|^2 \le \sup_{1\le j\le n}\|v_j\|^2\left\|\tau\sum_{j=1}^{n}(-A_N)^2E_N^{n-j+1}\right\|^2_{L(H)}\\ \le&\sup_{1\le j\le n}\|v_j\|^2\left(\tau\sum_{j=1}^{n}\lambda_{i,N}^2 r(\tau\lambda_{i,N}^2)^{n+1-j}\right)^2 \le C\sup_{1\le j\le n}\|v_j\|^2.\end{aligned} \tag{3.28}$$

∎

THEOREM 3.2. *Let Assumptions 2.1-2.2 be fulfilled and $\gamma \geq 2$. Then,*

$$\max_k \|A_N u_N^k\|_{L^p(\Omega;H)} < \infty, \ \max_k \|\Delta u_N^k\|_{L^p(\Omega;H)} < \infty \quad \forall p \geq 1. \tag{3.29}$$

Proof. At the beginning, let us prove an intermediate result.

$$\begin{aligned}
\mathbf{E} \sup_{0\leq k\leq M} |u_N^k|_1^p &\leq C\mathbf{E} \sup_{0\leq k\leq M} \|(-A_N)^{\frac{1}{2}} u_N^k\|^p \\
&\leq C\mathbf{E} \sup_{0\leq k\leq M} \|(-A_N)^{\frac{1}{2}} E_N^k P_N u_0\|^p \\
&\quad + C\mathbf{E} \sup_{0\leq k\leq M} \left\|\tau \sum_{j=1}^{k} (-A_N)^{\frac{3}{2}} E_N^{k-j+1} P_N P f(u_N^{j-1}))\right\|^p \\
&\quad + C\mathbf{E} \sup_{0\leq k\leq M} \left\|\sum_{j=1}^{k} (-A_N)^{\frac{1}{2}} E_N^{k-j+1} P_N g(u_N^{j-1}) \Delta W^j\right\|^p \\
&\leq C\|u_0\|_{L^p(\Omega;H^1)}^p + C\mathbf{E} \sup_{0\leq k\leq M} \left\|\tau \sum_{j=1}^{k} (-A_N)^{\frac{3}{2}} E_N^{k-j+1} P_N P f(u_N^{j-1}))\right\|^p \\
&\quad + C\left(\sum_{j=1}^{M-1} t_{k+1-j}^{-\frac{1}{2}} (\mathbf{E} \sup_{0\leq k\leq M} \|u_N^{j-1}\|^p)^{\frac{2}{p}} \tau\right)^{\frac{p}{2}}.
\end{aligned} \tag{3.30}$$

The first term is bounded by the assumption on initial condition and the third term is bounded by Theorem 3.1, and the second term is bounded by using (3.22).

$$\mathbf{E} \sup_{0\leq k\leq M} \left\|\tau \sum_{j=1}^{k} (-A_N)^{\frac{3}{2}} E_N^{k-j+1} P_N P f(u_N^{j-1}))\right\|^p \leq C\mathbf{E} \sup_{0\leq k\leq M} \left\|\sum_{j=0}^{k-1} (-A_N)^{\frac{1}{2}} f(u_N^{j-1})\tau\right\|^p. \tag{3.31}$$

Moreover, using the Gargliardo-Nirengberg inequality and (2.15), we have

$$\begin{aligned}
&\mathbf{E} \sup_{0\leq k\leq M} \left\|\sum_{j=0}^{k-1} (-A_N)^{\frac{1}{2}} f(u_N^j)\tau\right\|^p \\
&\leq \mathbf{E} \sup_{0\leq k\leq M} \left[\sum_{j=0}^{k-1} (\|(-A_N)^{\frac{1}{2}} u_N^j\| + \|(-A_N)^{\frac{1}{2}} (u_N^j)^3\|)\tau\right]^p \\
&\leq C\mathbf{E} \sup_{0\leq k\leq M} \left[\sum_{j=0}^{k-1} (\|\Delta u_N^j\| + \|(u_N^j)^2 \nabla u_N^j\|)\tau\right]^p \\
&\leq C\mathbf{E} \sup_{0\leq k\leq M} \left[\sum_{j=0}^{k-1} (\|\Delta u_N^j\| + \|u_N^j\|_{L^8}^2 \|\nabla u_N^j\|_{L^4})\tau\right]^p \\
&\leq C\mathbf{E} \sup_{0\leq k\leq M} \left[\sum_{j=0}^{k-1} (\|\Delta u_N^j\| + \|\Delta u_N^j\|^{\frac{3}{2}} \|u_N^j\|^{\frac{3}{2}})\tau\right]^p \\
&\leq C\mathbf{E} \left[\sum_{j=0}^{M} (\|\Delta u_N^j\|^2 + \|u_N^j\|^6)\tau\right]^p < C,
\end{aligned} \tag{3.32}$$

where we used (3.21) in the last step.

Therefore,

$$\mathbf{E}\sup_{0\le k\le M}|u_N^k|_1^p<\infty. \tag{3.33}$$

Now, we are about to prove the result. It is clear that

$$\begin{aligned}&\|A_Nu_N^{k+1}\|_{L^p(\Omega;H)}\\ =&\|A_NE_N^{k+1}u_N^0\|_{L^p(\Omega;H)}+\left\|\sum_{j=0}^{k}\frac{\tau A_N^2E_N^{k+1-j}}{1+\tau\|\Delta f(u_N^j)\|^2}P_Nf(u_N^j)\right\|_{L^p(\Omega;H)}\\ &+\left\|\sum_{j=0}^{k}A_NE_N^{k+1-j}P_Ng(u_N^j)\Delta W^j\right\|_{L^p(\Omega;H)}.\end{aligned} \tag{3.34}$$

The first term is easily bounded by

$$\|A_NE_N^{k+1}u_N^0\|_{L^p(\Omega;H)}\le\|u_0\|_{L^p(\Omega;H^2)} \tag{3.35}$$

and by using (3.33) and Assumption 2.1 (with $\nu=1$) and (2.5), we can estimate the third term.

$$\begin{aligned}&\mathbf{E}\left\|\sum_{j=0}^{k}A_NE^{k+1-j}P_Ng(u_N^j)\Delta W^j\right\|^p\le C\left(\sum_{j=0}^{k}(\mathbf{E}\|A_NE^{k+1-j}P_Ng(u_N^j)\Delta W^j\|^p)^{\frac{2}{p}}\right)^{\frac{p}{2}}\\ &\le C\left(\sum_{j=0}^{k}t_{k+1-j}^{-\frac12}(\mathbf{E}\|(-A_N)^{\frac12}g(u_N^j)\Delta W^j\|^p)^{\frac{2}{p}}\right)^{\frac{p}{2}}\le C\left(\sum_{j=0}^{k}t_{k+1-j}^{-\frac12}(\mathbf{E}\|g(u_N^j)\|_1^p)^{\frac{2}{p}}\tau\right)^{\frac{p}{2}}\\ &\le C\left(\sum_{j=0}^{k}t_{k+1-j}^{-\frac12}(\mathbf{E}\|u_N^j\|_1^p+C)^{\frac{2}{p}}\tau\right)^{\frac{p}{2}}<\infty.\end{aligned} \tag{3.36}$$

Using Theorem 3.1 and (3.23), the second term is bounded as follows:

$$\begin{aligned}&\tau\left\|\sum_{j=1}^{k}(-A_N)^2E_N^{n-j+1}P_NPf(u_N^{j-1})\right\|_{L^p(\Omega;H)}\le C(\mathbf{E}\sup_{0\le j\le k-1}\|f(u_N^j)\|^p)^{1/p}\\ &\le C\left(\mathbf{E}\sup_{0\le j\le k}\|u_N^j\|^p+\mathbf{E}\sup_{0\le j\le k}\|u_N^j\|_{L^{3p}}^{3p}\right)^{1/p}\le C\left(\mathbf{E}\sup_{0\le j\le k}\|u_N^j\|^p+\mathbf{E}\sup_{0\le j\le k}|u_N^j|_{H^1}^{3p}\right)^{1/p}<\infty.\end{aligned} \tag{3.37}$$

Consequently, $\max_k\|A_Nu_N^k\|_{L^p(\Omega;H)}<\infty$, which also implies

$$\begin{aligned}\max_k\|\Delta u_N^k\|_{L^p(\Omega;H)}=&\max_k\|P_N\Delta u_N^k\|_{L^p(\Omega;H)}=\max_k\|A_NR_Nu_N^k\|_{L^p(\Omega;H)}\\ =&\max_k\|A_Nu_N^k\|_{L^p(\Omega;H)}<\infty,\end{aligned} \tag{3.38}$$

where R_N is the Ritz projection operator and $A_NR_N=P_N\Delta$ (cf. [22, Page 11]). ∎

The third stability result can be proved now.

THEOREM 3.3. *Let $T>0$, $q\ge1$, $\gamma\ge2$. There exists a constant $C(T,q,\|u_0\|)$ such that*

$$\mathbf{E}\left[\sup_{0\le k\le M-1}\|u_N^k\|_\infty^q\right]\le C(T,q,\|u_0\|). \tag{3.39}$$

Proof. We use Agmon's inequality and Theorem 3.2 to derive that

$$\begin{aligned}&\mathbf{E}\sup_{0\le k\le M-1}\|u_N^k\|_\infty^q\le C\mathbf{E}\Big[\sup_{0\le k\le M-1}\|u_N^k\|^{\frac{q}{2}}\|\Delta u_N^k\|^{\frac{q}{2}}\Big]\\&\le C\mathbf{E}\sup_{0\le k\le M-1}\|u_N^k\|^q+C\mathbf{E}\sup_{0\le k\le M-1}\|\Delta u_N^k\|^q\le C(T,q,\|u_0\|).\end{aligned}\tag{3.40}$$

□

THEOREM 3.4. *Let Assumptions 2.1-2.2 be fulfilled and $\gamma\ge 2$. Then,*

$$\max_k\|\Delta f(u_N^k)\|_{L^p(\Omega;H)}^p\le C,\quad \forall p\ge 1.\tag{3.41}$$

Proof. By Theorem 3.3 and Theorem 3.2,

$$\begin{aligned}\mathbf{E}\|\Delta f(u_N^k)\|^p&\le\mathbf{E}\|(3u_N^k-1)\Delta u_N^k+6u_N^k|\nabla u_N^k|\|^p\\&\le\mathbf{E}[(3\|u_N^k\|_\infty^p+1)\|\Delta u_N^k\|^p+6\|u_N^k\|_\infty^p\|\nabla u_N^k\|^p]\\&\le C\mathbf{E}[(1+\|u_N^k\|_\infty^p)\|\Delta u_N^k\|^p]\\&\le C\sqrt{\mathbf{E}\|u_N^k\|_\infty^{2p}}\sqrt{\mathbf{E}\|\Delta u_N^k\|^{2p}}\le C.\end{aligned}\tag{3.42}$$

□

In addition, we need the following temporal continuity of $u(t)$.

LEMMA 3.2 ([5, Theorem 3.1]). *Assume Assumptions 2.1-2.2. Then,*

$$\|u(t)-u(s)\|_{L^p(\Omega;H)}\le C(t-s)^{\min\{\frac{1}{2},\frac{\gamma}{4}\}}.\tag{3.43}$$

To achieve our strong convergence result, we introduce an auxiliary process $\tilde u_N^n, 1\le n\le M$ (cf. [5,25]), defined by

$$\tilde u_N^{n+1}=\tilde u_N^n-\tau A_N^2\tilde u_N^{n+1}-\tau A_NP_Nf(u(t_n))+P_Ng(u(t_n))\Delta W^n,\tilde u_N^0=P_Nu_0.\tag{3.44}$$

It has the following stability properties.

THEOREM 3.5. *Let $\tilde u_N^n$ be defined as (3.44). Then,*

$$\begin{aligned}&\mathbf{E}[\sup_{0\le n\le M-1}\|\tilde u_N^n\|^p]<\infty,\\&\mathbf{E}[\sup_{0\le n\le M-1}\|\tilde u_N^n\|_\infty^p]<\infty,\\&\mathbf{E}\|A_N\tilde u_N^n\|^p<\infty,\\&\mathbf{E}\|A_Nf(\tilde u_N^n)\|^p<\infty.\end{aligned}$$

Proof. The proofs of the these four inequalities are almost the same as those of Theorems 3.1, 3.3, 3.2, and 3.4. So we omit them here. □

4. Error estimates

We now use the stability results established in the last section to carry out an error analysis.

LEMMA 4.1. *The following estimates hold:*

$$\|(E(t_j)-E_N^jP_N)v\|\le C(N^{-\beta}+\tau^{\min\{1,\frac{\beta}{4}\}})\|v\|_\beta,\quad v\in H^\beta.\tag{4.1}$$

Proof. The result is the counterpart of the estimate for FEM spatial approximation (cf. [13, Theorem 2.2]) by a slight modification and we omit it here. ∎

THEOREM 4.1. *Let $\gamma\geq 2$ and Assumptions 2.1-2.2 be fulfilled. Let $u(t)$ and $\tilde{u}_N^n$ be solutions of (1.2) and (3.44) respectively. Then, there exists a constant C independent of N and τ such that*

$$\|u(t_n)-\tilde{u}_N^n\|_{L_p(\Omega;H)}\leq C(N^{-\gamma}+\tau^{\frac{1}{2}}).$$

Proof. We follow along the lines of [25] to prove the result. Note that a similar auxiliary process introduced in [25] based upon fully-implicit scheme for stochastic Cahn-Hilliard equation with additive noise is

$$\tilde{u}_N^{n+1}=\tilde{u}_N^n-\tau A_N^2\tilde{u}_N^{n+1}-\tau A_N P_N f(u(t_{n+1}))+P_N\Delta W^n,\tilde{v}_N^0=P_N u_0. \tag{4.2}$$

Comparing (3.44) with (4.2), the proof of convergence rate is similar as that of [25] except that the following two additional terms emerging from the difference between our tamed scheme and full-implicit scheme used in the aforementioned work, and the difference between additive noise and multiplicative noise. For the difference of drift term, an application of Lemma 3.2 gives

$$\begin{aligned}
&\tau\Big\|\sum_{j=1}^{n}A_N E_N^{n-j+1}P_N(f(u(t_j)-f(u(t_{j-1}))))\Big\|_{L^p(\Omega;H)}\\
\leq& C\tau\sum_{j=1}^{n}t_{n-j+1}^{-\frac{3}{4}}\Big\|(1+\|u(t_j)\|_1^2+\|u(t_{j-1})\|_1^2)\|u(t_j)-u(t_{j-1})\|\Big\|_{L^p(\Omega;R)}\\
\leq& C\tau\sum_{j=1}^{n}t_{n-j+1}^{-\frac{3}{4}}(1+\|u(t_j)\|_{L^{2p}(\Omega;H^1)}^2+\|u(t_{j-1})\|_{L^{2p}(\Omega;H^1)}^2)\|u(t_j)-u(t_{j-1})\|_{L^{2p}(\Omega;H)}\\
\leq& C\tau^{\min\{\frac{1}{2},\frac{\gamma}{4}\}}
\end{aligned}\tag{4.3}$$

and for the stochastic term, we have by Lemma 2.1 and (4.1),

$$\begin{aligned}
&\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E(t_n-s)g(u(s))-E_N^{n-j+1}P_N g(u(t_{j-1}))dW(s)\Big\|_{L^p(\Omega;H)}\\
\leq&\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}E(t_n-s)(g(u(s))-g(u(t_{j-1}))dW(s)\Big\|_{L^p(\Omega;H)}\\
&+\Big\|\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}(E(t_n-s)-E_N^{n-j+1}P_N)g(u(t_{j-1}))dW(s)\Big\|_{L^p(\Omega;H)}\\
\leq& C\bigg(\sum_{j=1}^{n}\bigg(\mathbf{E}\Big\|\int_{t_{j-1}}^{t_j}E(t_n-s)(g(u(s))-g(u(t_{j-1})))dW^Q(s)\Big\|^p\bigg)^{\frac{2}{p}}\bigg)^{\frac{1}{2}}\\
&+C\bigg(\sum_{j=1}^{n}\bigg(\mathbf{E}\Big\|\int_{t_{j-1}}^{t_j}(E(t_n-s)-E_N^{n-j+1}P_N)g(u(t_{j-1}))dW^Q(s)\Big\|^p\bigg)^{\frac{2}{p}}\bigg)^{\frac{1}{2}}\\
\leq& C\bigg(\sum_{j=1}^{n}\bigg(\Big[\int_{t_{j-1}}^{t_j}\mathbf{E}\|g(u(s))-g(u(t_{j-1}))\|_{\mathcal{L}_2^0}^2 ds\Big]^{\frac{p}{2}}\bigg)^{\frac{2}{p}}\bigg)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+C\Bigg(\sum_{j=1}^{n}\Bigg(\Bigg[\int_{t_{j-1}}^{t_j}\mathbf{E}\|(E(t_n-s)-E_N^{n-j+1}P_N)g(u(t_{j-1}))\|_{\mathcal{L}_2^0}^2ds\Bigg]^{\frac{p}{2}}\Bigg)^{\frac{2}{p}}\Bigg)^{\frac{1}{2}}\\
\le& C\Bigg(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\|u(s)-u(t_{j-1})\|^2ds\Bigg)^{\frac{1}{2}}\\
&+C\Bigg(\sum_{j=1}^{n}\int_{t_{j-1}}^{t_j}\mathbf{E}\|(E(t_n-s)-E_N^{n-j+1}P_N)g(u(t_{j-1}))\|_{\mathcal{L}_2^0}^2ds\Bigg)^{\frac{1}{2}}\\
:=&\mathbb{A}_1+\mathbb{A}_2.
\end{aligned}
\tag{4.4}
$$

It is readily observed from Lemma 3.2 and (4.1) that

$$\mathbb{A}_1\le C\tau^{\min\{\frac{1}{2},\frac{\gamma}{4}\}}. \tag{4.5}$$

By Assumption 2.1, one has

$$\|g(u(t_j))\|_{L^p(\Omega;H^\gamma)}<\infty. \tag{4.6}$$

Hence, (4.1) implies

$$\mathbb{A}_2\le C(N^{-\gamma}+\tau^{\min\{\frac{1}{2},\frac{\gamma}{4}\}}). \tag{4.7}$$

Therefore, we can conclude that (cf. [5, 25])

$$
\begin{aligned}
\|\tilde{u}_N^{n+1}-u(t_{n+1})\|_{L^p(\Omega,H)}\le&\|\tilde{u}_N^{n+1}-u_N(t_{n+1})\|_{L^p(\Omega,H)}+\|u_N(t_{n+1})-u(t_{n+1})\|_{L^p(\Omega,H)}\\
\le& C(\tau^{\min\{\frac{1}{2},\frac{\gamma}{4}\}}+N^{-\gamma}).
\end{aligned}
\tag{4.8}
$$

□

For the convenience of proof, we define $e^n=u(t_n)-u_N^n, \tilde{e}^n=\tilde{u}_N^n-u_N^n, 1\le n\le M$.

THEOREM 4.2. *Let Assumptions 2.1-2.2 be fulfilled and $\gamma\ge 2$. Then,*

$$\mathbf{E}|\tilde{e}^{n+1}|_{-1}^{2p}+\mathbf{E}\Bigg[\tau\sum_{k=0}^{n}|\tilde{e}^{k+1}|_1^2\Bigg]^p\le C(N^{-2\gamma p}+\tau^{\min\{p,\frac{\gamma p}{2}\}}). \tag{4.9}$$

Proof. Note that

$$
\begin{aligned}
&\tilde{e}^{n+1}-\tilde{e}^n+\tau A_N^2\tilde{e}^{n+1}=-\tau A_NP_N\Bigg(f(u(t_n))-\frac{f(u_N^n)}{1+\tau\|\Delta f(u_N^n)\|^2}\Bigg)\\
&\qquad\qquad\qquad\qquad+P_N(g(u(t_n))-g(u_N^n))\Delta W^n,\\
&\tilde{e}_0=0.
\end{aligned}
\tag{4.10}
$$

Multiplying both sides by $2A_N^{-1}\tilde{e}^{n+1}$ yields

$$
\begin{aligned}
&\|\tilde{e}^{n+1}\|_{-1}^2-\|\tilde{e}^n\|_{-1}^2+\|\tilde{e}^{n+1}-\tilde{e}^n\|_{-1}^2+2\tau\|\tilde{e}^{n+1}\|_1^2\\
=&-2\tau\Bigg(f(u(t_n))-\frac{f(u_N^n)}{1+\tau\|\Delta f(u_N^n)\|^2},\tilde{e}^{n+1}\Bigg)+2(P_N(g(u(t_n))-g(u_N^n))\Delta W^n,A_N^{-1}\tilde{e}^{n+1})\\
:=&2\mathbb{A}_{-1}+2\mathbb{B}_{-1}.
\end{aligned}
\tag{4.11}
$$

The estimate of $\mathbb{A}_{-1}$ can be proved by following [8, Theorem 5.2]. Using Lemma 2.6 and the one-sided Lipschitz condition of $-f$:

$$-(f(u)-f(v),u-v)\le L\|u-v\|^2,\quad u,v\in L^6(\mathcal{O}),$$

we can derive

$$\begin{aligned}\mathbb{A}_{-1}=&-\tau(f(u(t_n))-f(\tilde{u}_N^n),\tilde{e}^{n+1})-\tau(f(\tilde{u}_N^n)-f(u_N^n),\tilde{e}^{n+1}-\tilde{e}^n)-\tau(f(\tilde{u}_N^n)-f(u_N^n),\tilde{e}^n)\\&-\frac{\tau^2\|\Delta f(u_N^n)\|^2}{1+\tau\|\Delta f(u_N^n)\|^2}(f(u_N^n),\tilde{e}^{n+1})\\ \le&\frac{\tau}{4}|\tilde{e}^{n+1}|_1^2+\tau|(-A_N)^{-\frac{1}{2}}(f(u(t_n))-f(\tilde{u}_N^n))|^2+\frac{1}{4}|\tilde{e}^{n+1}-\tilde{e}^n|_{-1}^2\\&+\tau^2|f(\tilde{u}_N^n)-f(u_N^n)|_1^2+L\tau\|\tilde{e}^n\|^2+\frac{\tau^2}{4}|\tilde{e}^{n+1}|_{-1}^2+\tau^2\|\Delta f(u_N^n)\|^4|f(u_N^n)|_1^2\\ \le&\frac{\tau}{4}|\tilde{e}^{n+1}|_1^2+C\tau(1+|u(t_n)|_1^4+|\tilde{u}_N^n|_1^4)\|u(t_n)-\tilde{u}_N^n\|^2+\frac{1}{4}|\tilde{e}^{n+1}-\tilde{e}^n|_{-1}^2+L\tau\|\tilde{e}^n\|^2\\&+\frac{\tau^2}{4}|\tilde{e}^{n+1}|_{-1}^2+\tau^2\|\Delta f(u_N^n)\|^4|f(u_N^n)|_1^2+\tau^2|f(\tilde{u}_N^n)-f(u_N^n)|_1^2.\end{aligned}\tag{4.12}$$

For $\mathbb{B}_{-1}$, we have

$$\begin{aligned}\mathbb{B}_{-1}=&-((-A_N)^{-\frac{1}{2}}P_N[g(u(t_n))-g(u_N^n)\Delta W^n],(-A_N)^{-\frac{1}{2}}(\tilde{e}^{n+1}-\tilde{e}^n))\\&-((-A_N)^{-\frac{1}{2}}P_N[g(u(t_n))-g(u_N^n)\Delta W^n],(-A_N)^{-\frac{1}{2}}\tilde{e}^n)\\ \le&\frac{1}{2}|\tilde{e}^{n+1}-\tilde{e}^n|_{-1}^2+\frac{1}{2}\|(-A_N)^{-\frac{1}{2}}P_N(g(u(t_n))-g(u_N^n)\Delta W^n)\|^2\\&+((-A_N)^{-\frac{1}{2}}P_N[g(u(t_n))-g(u_N^n)\Delta W^n],(-A_N)^{-\frac{1}{2}}\tilde{e}^n).\end{aligned}\tag{4.13}$$

Therefore, rearranging these terms and using the estimate $\|\tilde{e}^n\|^2\le C|\tilde{e}^n|_{-1}|\tilde{e}^n|_1$ lead to

$$\begin{aligned}&(1-\frac{\tau^2}{4})|\tilde{e}^{n+1}|_{-1}^2+\frac{1}{4}|\tilde{e}^{n+1}-\tilde{e}^n|_{-1}^2+\frac{7\tau}{4}|\tilde{e}^{n+1}|_1^2\\ \le&(1+C\tau)|\tilde{e}^n|_{-1}^2+C\tau[(1+|u(t_n)|_1^4+|\tilde{u}_N^n|_1^4)\|u(t_n)-\tilde{u}_N^n\|^2]+\tau^2[\|\Delta f(u_N^n)\|^4|f(u_N^n)|_1^2]\\&+\tau^2|f(\tilde{u}_N^n)-f(u_N^n)|_1^2+\|(-A_N)^{-\frac{1}{2}}P_N(g(u(t_n))-g(u_N^n)\Delta W^n)\|^2\\&+((-A_N)^{-\frac{1}{2}}P_N[g(u(t_n))-g(u_N^n)\Delta W^n],(-A_N)^{-\frac{1}{2}}\tilde{e}^n)+\frac{\tau}{2}|\tilde{e}^n|_1^2.\end{aligned}\tag{4.14}$$

Applying Lemma 2.4 with $A_n^2=|\tilde{e}^n|_{-1}^2$ and $B_n^2=|\tilde{e}^n|_1^2$ gives

$$\begin{aligned}|\tilde{e}^{n+1}|_{-1}^2+\tau\sum_{k=0}^n|\tilde{e}^{k+1}|_1^2\le&\tau\sum_{k=0}^n(1+C\tau)^{n-k}(1+|u(t_k)|_1^4+|\tilde{u}_N^k|_1^4)\|u(t_k)-\tilde{u}_N^k\|^2\\&+\tau^2\sum_{k=0}^n(1+C\tau)^{n-k}(\|\Delta f(u_N^k)\|^4|f(u_N^k)|_1^2+|f(\tilde{u}_N^k)-f(u_N^k)|_1^2)\\&+\sum_{k=0}^n(1+C\tau)^{n-k}((-A_N)^{-\frac{1}{2}}P_N[g(u(t_k))-g(u_N^k)\Delta W^k],(-A_N)^{-\frac{1}{2}}\tilde{e}^k)\\&+\sum_{k=0}^n(1+C\tau)^{n-k}\|(-A_N)^{-\frac{1}{2}}P_N(g(u(t_k))-g(u_N^k)\Delta W^k)\|^2.\end{aligned}\tag{4.15}$$

A simple computation implies that

$$(1+C\tau)^n\le e^{Cn\tau}\le e^{CT},\ n\le M.\tag{4.16}$$

Hence, we take p-th power on both sides and then expectation to have

$$
\begin{aligned}
&\mathbf{E}|\tilde{e}^{n+1}|_{-1}^{2p}+\mathbf{E}\Big[\tau\sum_{k=0}^{n}|\tilde{e}^{k+1}|_1^2\Big]^p \le C\mathbf{E}\Big(\tau\sum_{k=0}^{n}(1+|u(t_k)|_1^4+|\tilde{u}_N^k|_1^4)\|u(t_k)-\tilde{u}_N^k\|^2\Big)^p\\
&\qquad +C\mathbf{E}\Big(\tau^2\sum_{k=0}^{n}(\|\Delta f(u_N^k)\|^4|f(u_N^k)|_1^2+|f(\tilde{u}_N^k)-f(u_N^k)|_1^2)\Big)^p\\
&\qquad +C\mathbf{E}\Big(\sum_{k=0}^{n}((-A_N)^{-\frac{1}{2}}P_N[g(u(t_k))-g(u_N^k)\Delta W^k],(-A_N)^{-\frac{1}{2}}\tilde{e}^k)\Big)^p\\
&\qquad +C\mathbf{E}\Big(\sum_{k=0}^{n}(\|(-A_N)^{-\frac{1}{2}}P_N(g(u(t_k))-g(u_N^k))\Delta W^k\|^2\Big)^p\\
&\qquad := I_1+I_2+I_3+I_4.
\end{aligned}
\tag{4.17}
$$

By Cauchy-Schwartz inequality and Theorem 4.1, I_1 can be estimated as follows.

$$
\begin{aligned}
I_1 &\le C\sqrt{\mathbf{E}\Big[\tau\sum_{k=0}^{n}(1+|u(t_k)|_1^4+|\tilde{u}_N^k|_1^4)^2\Big]^p}\sqrt{\mathbf{E}\Big[\tau\sum_{k=0}^{n}\|u(t_k)-\tilde{u}_N^k\|^4\Big]^p}\\
&\le C(N^{-2\gamma p}+\tau^{\min\{p,\frac{\gamma}{2}p\}}).
\end{aligned}
\tag{4.18}
$$

I_2 can be easily bounded by using our stability results Lemma 3.2, Lemma 3.4 and Lemma 3.5.

$$
I_2 \le C\tau^p. \tag{4.19}
$$

Appealing to Lemma 2.1 and the elementary inequality $(x+y)^{\frac{2}{p}}\le x^{\frac{2}{p}}+y^{\frac{2}{p}}, x,y\ge 0, \forall p\ge 2$,

$$
\begin{aligned}
I_3 &\le C\Big(\sum_{j=0}^{n}[\mathbf{E}(|(-A_N)^{-\frac{1}{2}}P_N(g(u(t_j)-g(u_N^j))\Delta W^j,(-A_N)^{-\frac{1}{2}}\tilde{e}^j)|^p]^{\frac{2}{p}}\Big)^{\frac{p}{2}}\\
&\le C\Big(\sum_{j=0}^{n}\{\mathbf{E}[(\|g(u(t_j)-g(u_N^j))\|_{-1}^4+\|\tilde{e}_N^j\|_{-1}^4)\tau]^{\frac{p}{2}}\}^{\frac{2}{p}}\Big)^{\frac{p}{2}}\\
&\le C\Big(\sum_{j=0}^{n}\{\mathbf{E}[(\|u(t_j)-\tilde{u}_N^j)\|_{-1}^4+\|\tilde{e}_N^j\|_{-1}^4)\tau]^{\frac{p}{2}}\}^{\frac{2}{p}}\Big)^{\frac{p}{2}}\\
&\le C\Big(\tau\sum_{j=0}^{n}\{\mathbf{E}\|u(t_j)-\tilde{u}_N^j)\|_{-1}^{2p}+\mathbf{E}\|\tilde{e}_N^j\|_{-1}^{2p}\}^{\frac{2}{p}}\Big)^{\frac{p}{2}}\\
&\le C\Big(\tau\sum_{j=0}^{n}(\mathbf{E}\|u(t_j)-\tilde{u}_N^j)\|_{-1}^{2p})^{\frac{2}{p}}+(\mathbf{E}\|\tilde{e}_N^j\|_{-1}^{2p})^{\frac{2}{p}}\Big)^{\frac{p}{2}}\\
&\le C\tau\sum_{j=0}^{n}\mathbf{E}\|u(t_j)-\tilde{u}_N^j)\|_{-1}^{2p}+C\tau\sum_{j=0}^{n}\mathbf{E}\|\tilde{e}_N^j\|_{-1}^{2p}\\
&\le C(N^{-2\gamma p}+\tau^{\min\{p,\frac{\gamma p}{2}\}})+C\tau\sum_{j=0}^{n}\mathbf{E}\|\tilde{e}_N^j\|_{-1}^{2p}.
\end{aligned}
\tag{4.20}
$$

Similarly, by the standard Burkholder-Davis-Gundy inequality [6, Page 114], Lemma 2.5 and (2.5), we have

$$\begin{aligned}
I_4 &\le n^{p-1}\sum_{k=0}^{n}\mathbf{E}\|(-A_N)^{-\frac{1}{2}}P_N(g(u(t_k))-g(u_N^k))\Delta W^k)\|^{2p}\\
&\le C\tau\sum_{k=0}^{n}\mathbf{E}\|(-A_N)^{-\frac{1}{2}}P_N(g(u(t_k))-g(u_N^k))\|_{\mathcal{L}_2^0}^{2p}\\
&\le C\tau\sum_{k=0}^{n}\mathbf{E}\|(-A_N)^{-\frac{1}{2}}P_N(g(u(t_k))-g(u_N^k))\|^{2p}\\
&\le C\tau\sum_{k=0}^{n}\mathbf{E}\|(g(u(t_k))-g(u_N^k))\|_{-1}^{2p}\\
&\le C\tau\sum_{k=0}^{n}\mathbf{E}\|u(t_k)-u_N^k\|_{-1}^{2p}\\
&\le C(N^{-2\gamma p}+\tau^{\min\{p,\frac{\gamma p}{2}\}})+C\tau\sum_{j=0}^{n}\mathbf{E}\|\tilde{e}_N^j\|_{-1}^{2p}. \qquad (4.21)
\end{aligned}$$

Combining the estimates of I_1-I_4, and substituting them into (4.17) leads to

$$\mathbf{E}|\tilde{e}^{n+1}|_{-1}^{2p}+\mathbf{E}\left[\tau\sum_{k=0}^{n}|\tilde{e}^{k+1}|_1^2\right]^p\le C(N^{-2\gamma p}+\tau^{\min\{p,\frac{\gamma p}{2}\}})+C\tau\sum_{j=0}^{n}\mathbf{E}\|\tilde{e}^j\|_{-1}^{2p}. \qquad (4.22)$$

The discrete Gronwall inequality implies the desired result. ∎

THEOREM 4.3. *Let Assumptions 2.1-2.2 be fulfilled and $\gamma\ge 2$ and $u(t)$ and u_N^n be solutions of (1.2) and (3.3) respectively. Then, there exists a constant C independent of N and M such that*

$$\|u(t_n)-u_N^n\|_{L_2(\Omega;H)}\le C(\tau^{\frac{1}{2}}+N^{-\gamma}),\quad t>0. \qquad (4.23)$$

Proof. Clearly,

$$\|u_N^{n+1}-u(t_{n+1})\|_{L^2(\Omega,H)}\le\|\tilde{u}_N^{n+1}-u(t_{n+1})\|_{L^2(\Omega,H)}+\|u_N^{n+1}-\tilde{u}_N^{n+1}\|_{L^2(\Omega,H)}, \qquad (4.24)$$

and we only need to bound the second term.

Recall from (4.10) that

$$\begin{aligned}
\tilde{e}^{n+1}-\tilde{e}^n+\tau A_N^2\tilde{e}^{n+1}&=\tau A_NP_N\left(f(u(t_n))-\frac{f(u_N^n)}{1+\tau\|\Delta f(u_N^n)\|^2}\right)\\
&\qquad+P_N(g(u(t_n))-g(u_N^n))\Delta W^n,\\
\tilde{e}_0&=0.
\end{aligned}$$

We rewrite it in equivalent form

$$\begin{aligned}
\tilde{e}^{n+1}=&\tau\sum_{k=0}^{n}A_NE_N^{n-k+1}P_N\left(f(u(t_k))-\frac{f(u_N^k)}{1+\tau\|\Delta f(u_N^k)\|^2}\right)\\
&+\sum_{k=0}^{n}E_N^{n-k+1}P_N(g(u(t_k))-g(u_N^k))\Delta W^k.
\end{aligned}$$

Hence,

$$
\begin{aligned}
\|\tilde{e}^{n+1}\|_{L^2(\Omega;H)} \le & \tau\sum_{k=0}^{n}\|A_N E_N^{n-k+1}P_N(f(u(t_k))-f(\tilde{u}_N^k))\|_{L^2(\Omega;H)} \\
& +\tau\sum_{k=0}^{n}\|A_N E_N^{n-k+1}P_N(f(\tilde{u}_N^k)-f(u_N^k))\|_{L^2(\Omega;H)} \\
& +\tau^2\sum_{k=0}^{n}\|A_N E_N^{n-k+1}P_N f(u_N^k)\|\Delta f(u_N^k)\|^2\|_{L^2(\Omega;H)} \\
& +\sum_{k=0}^{n}\|E_N^{n-k+1}P_N(g(u(t_k))-g(u_N^k))\Delta W^k\|_{L^2(\Omega;H)} \\
:= & \sum_{i=1}^{4} I_i.
\end{aligned} \tag{4.25}
$$

We estimate these terms in turn.

$$
\begin{aligned}
I_1 & \le C\tau\sum_{k=0}^{n} t_{n-k+1}^{-\frac{1}{2}}\|f(u(t_k))-f(\tilde{u}_N^k)\|_{L^2(\Omega;H)} \\
& \le C\tau\sum_{k=0}^{n} t_{n-k+1}^{-\frac{1}{2}}\|u(t_k)-\tilde{u}_N^k\|_{L^4(\Omega;H)}(1+\|u(t_k)\|^2_{L^8(\Omega;L^\infty)}+\|\tilde{u}_N^k\|^2_{L^8(\Omega;L^\infty)}) \\
& \le C(N^{-\gamma}+\tau^{\min\{\frac{1}{2},\frac{\gamma}{4}\}}).
\end{aligned}
$$

We follow along the lines of [25] to prove I_2 by modifying some arguments. Using the result of Theorem 4.2, (2.22) and our stability results,

$$
\begin{aligned}
I_2 & \le \left\|\tau\sum_{k=0}^{n} t_{n-k+1}^{-\frac{1}{4}}\|(-A_N)^{\frac{1}{2}}P_N P(f(\tilde{u}_N^k)-f(u_N^k))\|\right\|_{L^2(\Omega;R)} \\
& \le \left\|\tau\sum_{k=0}^{n} t_{n-k+1}^{-\frac{1}{4}}|\tilde{e}^k|_1(1+|\tilde{u}_N^k|_1^2+|u_N^k|_1^2+\|\tilde{u}_N^k\|_\infty^2+\|u_N^k\|_\infty^2)\right\|_{L^2(\Omega;R)} \\
& \le \left\|\left(\tau\sum_{k=0}^{n}|\tilde{e}^k|_1^2\right)^{\frac{1}{2}}\left(\tau\sum_{k=0}^{n} t_{n-k+1}^{-\frac{1}{2}}(1+|\tilde{u}_N^k|_1^4+|u_N^k|_1^4+\|\tilde{u}_N^k\|_\infty^4+\|u_N^k\|_\infty^4)\right)^{\frac{1}{2}}\right\|_{L^2(\Omega;R)} \\
& \le \left\|\tau\sum_{k=0}^{n}|\tilde{e}^k|_1^2\right\|_{L^2(\Omega;R)}^{\frac{1}{2}}\left\|\tau\sum_{k=0}^{n} t_{n-k+1}^{-\frac{1}{2}}(1+|\tilde{u}_N^k|_1^4+|u_N^k|_1^4+\|\tilde{u}_N^k\|_\infty^4+\|u_N^k\|_\infty^4)\right\|_{L^2(\Omega;R)}^{\frac{1}{2}} \\
& \le C(\tau^{\min\{\frac{1}{2},\frac{\gamma}{4}\}}+N^{-\gamma}).
\end{aligned}
$$

and

$$
I_3 \le \tau^2\sum_{k=0}^{n} t_{n-k+1}^{-\frac{1}{2}}\|f(u_N^k)\|_{L^4(\Omega;H)}\|\Delta f(u_N^k)\|^2_{L^8(\Omega;H)} \le C\tau.
$$

Finally, by the Ito isometry and (2.5),

$$
I_4^2 = \mathbf{E}\left\|\sum_{k=0}^{n}\int_{t_{k-1}}^{t_k} E^{n-k+1}P_N\big(g(u(t_k))-g(u_N^k)\big)dW^Q(s)\right\|^2
$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{n}\int_{t_k}^{t_{k+1}} \mathbf{E}\|u(t_k)-u_N^k\|^2 d\sigma\\
&\leq C(\tau^{\min\{1,\frac{\gamma}{2}\}}+N^{-2\gamma})+C\tau\sum_{k=0}^{n}\mathbf{E}\|\tilde{e}^k\|^2. \qquad (4.26)
\end{aligned}
$$

Gathering the above estimates and substituting into (4.25), we obtain

$$
\mathbf{E}\|\tilde{e}^{n+1}\|^2 \leq C(\tau^{\min\{1,\frac{\gamma}{2}\}}+N^{-2\gamma})+C\tau\sum_{k=0}^{n}\mathbf{E}\|\tilde{e}^k\|^2. \qquad (4.27)
$$

Then, the discrete Gronwall inequality gives the result. □

5. Efficient implementation and numerical results

We shall describe how to implement the full discretization efficiently, and present several numerical results to validate the scheme and theoretical estimates.

5.1. Efficient implementation.

We describe below our polynomial based full discretization algorithm. A key observation from our semi-discretization is that the stiff matrix and mass matrix resulted can be diagonalized simultaneously (cf. [21, Chapter 8]). To fix the idea, we set $\mathcal{O}=[-1,1]^2$.

Let $L_m(x)$ be the Legendre polynomials on $[-1,1]$ and

$$
\phi_m(x)=\frac{1}{\sqrt{-b_m(4m+6)}}(L_m(x)+b_mL_{m+2}(x)),\quad m\geq 0,
$$

where $b_m=-m(m+1)/((m+2)(m+3))$. Then, [21, Chapter 4] gives $(\phi_i',\phi_j')=\delta_{ij}$ and $\frac{\partial\phi_m}{\partial x}\big|_{x=\pm1}=0$. Furthermore, the mass matrix $M=m_{ij}=(\phi_i,\phi_j)$ is penta-diagonal whose non-zero elements are

$$
m_{ij}=m_{ji}=\begin{cases}\dfrac{2}{2i+1}+b_i^2\dfrac{2}{2i+5}, & j=i,\\[2mm] b_i^2\dfrac{2}{2i+5}, & j=i+2.\end{cases} \qquad (5.1)
$$

Take the eigen-decomposition of M,

$$
ME=E\Lambda
$$

with $E=(e_{jk})$, $\Lambda=\mathrm{diag}(\lambda_0,\lambda_1,\cdots,\lambda_{N-2})$. Then, we construct basis functions

$$
\psi_k(x)=\sum_{j=0}^{N-2}e_{jk}\phi_j(x), \qquad (5.2)
$$

which have powerful properties:

$$
(\psi_k',\psi_j')=\delta_{kj},\quad (\psi_k,\psi_j)=\lambda_j\delta_{kj}.
$$

Let us denote

$$
u_N^{n+1}=\sum_{m,k=0}^{N-2}c_{mk}^{n+1}\psi_m(x)\psi_k(y),\ \mathbf{C}^{n+1}=(c_{mk}^{n+1})_{m,k=0,1,\cdots,N-2};
$$

$$\mu_N^{n+1}=\sum_{m,k=0}^{N-2} d_{mn}^{n+1}\psi_m(x)\psi_k(y),\, \mathbf{D}^{n+1}=(d_{mk}^{n+1})_{m,k=0,1,\cdots,N-2};$$
$$f_{mk}^n=\int_{\mathcal{O}} f(u_N^n)\psi_m(x)\psi_k(y)dxdy,\, \mathbf{F}^n=(f_{mk})_{m,k=0,1,\cdots,N-2}^n;$$
$$g_{mk}^{j_1j_2}=\int_{\mathcal{O}} g(u_N^n)e_{j_1j_2}(x,y)\psi_m(x)\psi_k(y)dxdy,\, \mathbf{G}_{j_1j_2}^n=(g_{m,k}^{j_1j_2})_{m,k=0,1,\cdots,N-2}. \tag{5.3}$$

Taking $\phi=\psi=\psi_i(x)\psi_j(y), i,j=0,\cdots,N-2$ in (3.1b) leads to

$$\begin{cases}\Lambda(\mathbf{C}^{n+1}-\mathbf{C}^n)\Lambda=-\tau(\Lambda\mathbf{D}^{n+1}+\mathbf{D}^{n+1}\Lambda)+\sum_{j_1,j_2=1}^{J}\sqrt{q_{j_1j_2}}\mathbf{G}_{j_1j_2}^n\Delta\beta_{j_1j_2}^n,\\ \Lambda\mathbf{D}^{n+1}\Lambda=(\Lambda\mathbf{C}^{n+1}+\mathbf{C}^{n+1}\Lambda)+\dfrac{1}{(1+\tau\|\Delta f(u_N^n)\|^2)}\mathbf{F}^n.\end{cases} \tag{5.4}$$

For convenience of computation, we follow the spirit of [21, Chapter 8] and rewrite (5.4) as

$$\begin{cases}\Lambda\otimes\Lambda(\mathbf{u}^{n+1}-\mathbf{u}^n)=-\tau(\Lambda\otimes I+I\otimes\Lambda)\mu^{n+1}+\sum_{j_1,j_2=1}^{J}\sqrt{q_{j_1j_2}}\widetilde{\mathbf{g}}_{j_1j_2}^n\Delta\beta_{j_1j_2}^n,\\ \Lambda\otimes\Lambda\mu^{n+1}=(\Lambda\otimes I+I\otimes\Lambda)\mathbf{u}^{n+1}+\dfrac{1}{(1+\tau\|\Delta f(u_N^n)\|^2)}\widetilde{\mathbf{f}}^n,\end{cases} \tag{5.5}$$

where $\mathbf{u}^n$ and μ^n are vectors of length $(N-1)^2$ formed by columns of $\mathbf{C}^n$ and $\mathbf{D}^n$ respectively, and $\widetilde{\mathbf{f}}^n$ and $\widetilde{\mathbf{g}}_{j_1j_2}^n$ are vectors formed by $\mathbf{F}^n$ and $\mathbf{G}_{j_1j_2}^n$, and $\Delta\beta_{j_1j_2}^n\sim N(0,\tau)$. The system (5.5) is a diagonal system for $(\mathbf{u}^{n+1},\mu^{n+1})$ so it can be easily solved. The main computational costs are the evaluation of $\mathbf{F}^n$ and $\mathbf{G}_{j_1j_2}^n$ which can be efficiently approximated by the Gauss-Legendre quadrature.

5.2. Numerical experiments. In this subsection, three numerical experiments are provided to illustrate the theoretical results claimed in the previous sections.

EXAMPLE 5.1. *Consider the following 1-d Cahn-Hilliard-Cook equation on the time domain* $0\le t\le T=0.1$*:*

$$\begin{cases}du=\dfrac{\partial^2\mu}{\partial x^2}dt+dW^Q(t),\ \dfrac{\partial u}{\partial x}\Big|_{x=0,1}=0,\ x\in I=(0,1),\\ \mu=-\dfrac{\partial^2 u}{\partial x^2}+(u^3-u),\ \dfrac{\partial \mu}{\partial x}\Big|_{x=0,1}=0,\\ u(0,\cdot)=\cos\pi x\end{cases}$$

and we take

$$W^Q(t)=\sum_{j=1}^{\infty}\sqrt{q_j}\cos(j\pi x)\beta_j(t).$$

To measure the spatial error, we run $K=1024$ independent realizations for each spatial expansion terms with $N=12,14,\cdots,20$ and temporal steps $\tau=T/2^{21}$ and truncate the first 100 terms in $W^Q(t)$. Noting that the noise is additive, we can avoid the expensive numerical quadratures for $\mathbf{G}_{j_1j_2}^n$ in (5.3) by using the following identities (cf. [19, Page 433]):

$$\int_0^a \sin bx L_{2n+1}(x/a)dx=(-1)^n\sqrt{\frac{\pi a}{2b}}J_{2n+\frac{3}{2}}(ab),$$
$$\int_0^a \cos bx L_{2n}(x/a)dx=(-1)^n\sqrt{\frac{\pi a}{2b}}J_{2n+\frac{1}{2}}(ab),$$

where J is the Bessel function of the first kind.

Since the true solution is unknown, we take the numerical approximation when $\tau = T/2^{21}$ and $N = 100$ as a surrogation. Moreover, the error $E\|u_N^M - u(0.1)\|$ is approximated by

$$E\|u(T) - u_N^M\| \approx \sqrt{\frac{1}{K}\sum_{k=1}^{K} \|u_N^M(\omega_i) - u(t_m,\omega_i)\|^2}. \tag{5.6}$$

In this case, A and Q commute and we examine the condition $\|(-A)^{\frac{\gamma-2}{2}} Q^{\frac{1}{2}}\|_{HS} < \infty$ (cf. [25]) and consider the following two cases $Q = (-A)^{-1.5005}$ and $Q = (-A)^{-3.5005}$ respectively, which are associated with $\gamma = 3$ and 5. One can observe from Figure 5.1 that the spatial error decays $\mathcal{O}(N^{-4})$ and $\mathcal{O}(N^{-5})$ for these two cases, which agrees with our theorem prediction. Similarly, In order to find the temporal error convergence rate, we froze $N = 100$ and take time steps $\tau = 2^7, 2^8, \cdots, 2^{11}$ and truncate the first 100 terms in $W^Q(t)$. A surrogate of true solution is obtained for $N = 100$ and $\tau = T/2^{21}$. Figure 5.2 demonstrates that the temporal error decays at a rate about $\mathcal{O}(\tau)$.

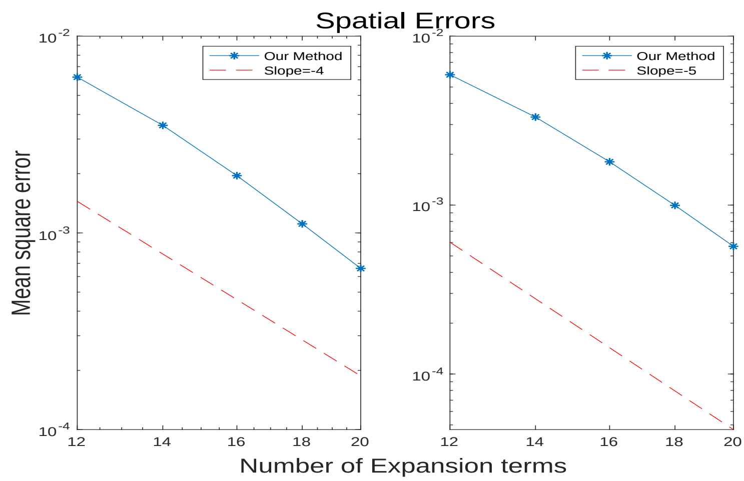

FIG. 5.1. *Spatial errors of 1-d Cahn-Hilliard-Cook equation with (left): $q_j = j^{-3.001}$ and (right) $q_j = j^{-7.001}$.*

It is well-known that a temporal strong convergence rate of $\mathcal{O}(\tau)$ can be obtained by using Euler's method for SDEs. One may also obtain this rate by Euler's method for some SPDEs [23]. In this example, we can also observe this phenomenon for CHC equation.

EXAMPLE 5.2. *Next, we consider the 1-d stochastic Cahn-Hilliard equation with multiplicative noise:*

$$\begin{cases} du = \dfrac{\partial^2 \mu}{\partial x^2} dt + \dfrac{1-u^2}{1+u^2} dW^Q(t), \ \dfrac{\partial u}{\partial x}\Big|_{x=0,1} = 0, \ x \in I = (0,1), \\ \mu = -\dfrac{\partial^2 u}{\partial x^2} + (u^3 - u), \ \dfrac{\partial \mu}{\partial x}\Big|_{x=0,1} = 0, \\ u(0,\cdot) = \cos \pi x \end{cases}$$

and the remaining settings are exactly the same as those in Example 5.1.

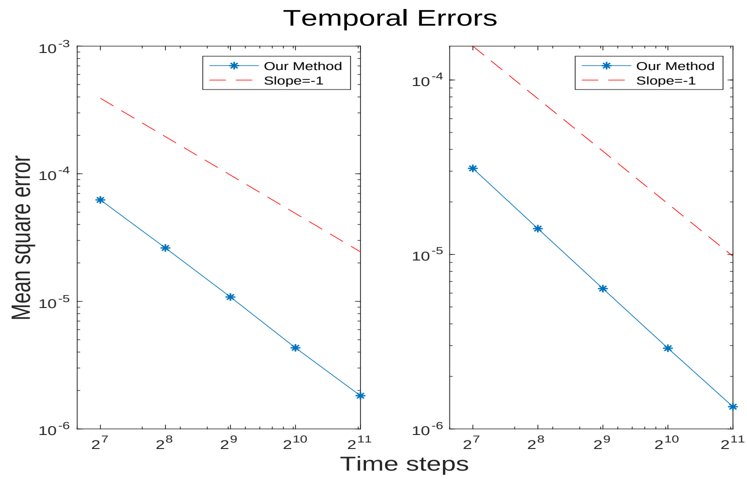

FIG. 5.2. *Temporal errors of 1-d Cahn-Hilliard-Cook equation with (left):* $q_j = j^{-3.001}$ *and (right):* $q_j = j^{-7.001}$.

We choose $K = 192$ and $Q = (-A)^{-1.5005}$, and $N = 100, \tau = T/2^{18}$ as parameters for surrogation of true solution since numerical quadrature for $\mathbf{G}^n_{j_1 j_2}$ of (5.3) has to be performed in this case. From Figure 5.3, one clearly observes a rate of $\mathcal{O}(\tau^{3/4})$ and $\mathcal{O}(N^{-2})$ for temporal and spatial errors, respectively, which are slightly better than our prediction of Theorem 4.3.

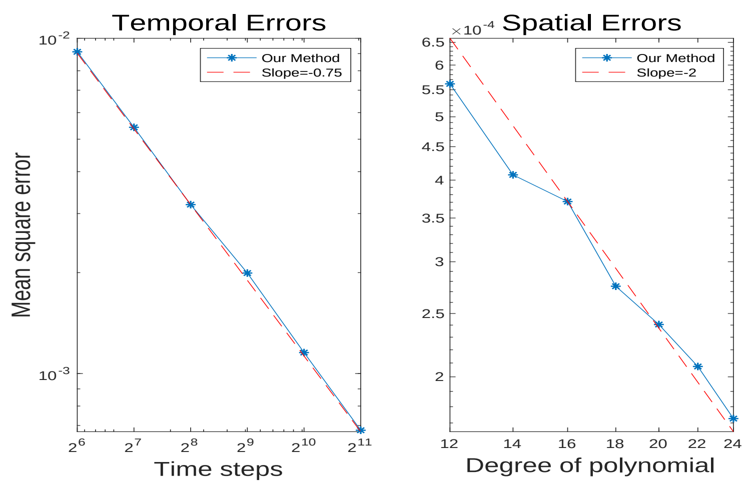

FIG. 5.3. *Temporal and Spatial errors of 1-d stochastic Cahn-Hilliard equation with multiplicative noise.*

EXAMPLE 5.3. *Consider the following 2-d stochastic Cahn-Hilliard equation:*

$$d\phi = \kappa_1 \Delta \mu dt + \kappa_2 g(\phi) dW^Q(t), \quad \left.\frac{\partial \mu}{\partial n}\right|_{\partial\Omega} = 0$$

$$\mu = -\epsilon \Delta \phi + \frac{1}{\epsilon} F'(\phi), \quad \left.\frac{\partial \phi}{\partial n}\right|_{\partial\Omega} = 0$$

$$\phi(0)=\phi_0, \tag{5.7}$$

where $\epsilon=1,\kappa_1=1/40,\ \kappa_2=20,\ g(\phi)=\sin(\phi), 0\le t\le T=0.1$ *and*

$$W^Q(t)=\sum_{j_1,j_2=1}^{10} 1/(j_1^2+j_2^2)\cos(j_1\pi x)\cos(j_2\pi y)\beta_{j_1,j_2}(t).$$

In the experiment, we choose $K=192$ in (5.6) to measure the error. In order to find the spatial convergence rate, we choose the approximation for $N=50$ and $\tau=T/2048$ as a surrogate of true solution for spatial convergence rate. From Figure 5.3, we can clearly observe a spatial convergence rate of approximately $\mathcal{O}(N^{-3.5})$ for $N=7,9,11,13$ and 15. Similarly, we froze the numerical approximation of $N=50$ and $M=2048$ as a surrogate of true solution and temporal convergence rate $\mathcal{O}(\sqrt{\tau})$ for $\tau=1/2^4,1/2^5,\cdots,1/2^9$ can be observed from Figure 5.4.

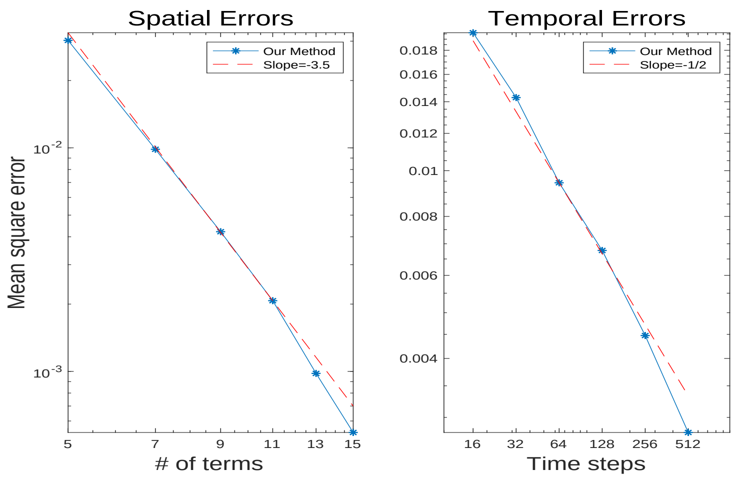

FIG. 5.4. *Numerical errors of 2d stochastic C-H equation.*

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